Enumeration of some doubly semi-equivelar maps on torus

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May 4, 2020

Abstract

The 2-uniform tilings of the plane provide doubly semi-equivelar maps on torus, as the 1-uniform tilings provide semi-equivelar maps. There are twenty distinct 2-uniform tilings of the plane. In this article, we give a construction to classify and enumerate doubly semi-equivelar maps on the surface of torus corresponding to the 2-uniform tilings $[3^6 : 3^3.4^2]_1$, $[3^6 : 3^3.4^2]_2$, $[3^6 : 3^2.4.3.4]$, $[3^3.4^2 : 3^2.4.3.4]_1$, $[3^3.4^2 : 3^2.4.3.4]_2$, $[3^3.4^2 : 4^4]_1$ and $[3^3.4^2 : 4^4]_2$.

Keywords: 2-uniform tilings, Doubly semi-equivelar maps, Torus.
MSC(2010): 52B70.

1 Introduction

A tiling of the plane, using regular polygons, is called a k-uniform tiling if it is edge to edge and has precisely k classes of vertices under the action of its symmetry. Grünbaum [9] described such tilings nicely in his book "Tilings and Patterns". Since the plane is the universal cover of torus and Klein bottle, the 2-uniform tilings suggest to explore the same type (locally) maps on these two surfaces as associated to these tilings. In this context, an extensive study has been done for the eleven 1-uniform tilings under the names regular, degree-regular, equivelar and semi-equivelar maps. Altshuler [1], Kurth [13], Datta and Nilakantan [7], Datta and Upadhyay [8], Brehm and Kühnel[5] studied semi-equivelar maps corresponding to the 1-uniform tilings $[3^6]$, $[4^4]$ and $[6^3]$. Tiwari and Upadhyay [10], Maity and Upadhyay [14] studied semi-equivelar maps on torus and Klein bottle corresponding to the tilings $[3^4.6]$, $[3^3.4^2]$, $[3^2.4.3.4]$, $[3.6.3.6]$, $[3.4.6.4]$, $[3.12^2]$, $[4.6.12]$ and $[4.8^2]$. More recently, Datta and Maity [6] studied such maps in a group theocrtical point of view.

In [11], Tiwari et al. introduced a theory of maps on torus and Klein bottle corresponding to the 2-uniform tiings under the name doubly semi-equivelar maps. They enumerated all the doubly semi-equivelar maps on these two surfaces up to 15 vertices corresponding to the 2-uniform tilings $[3^6 : 3^3.4^2]_1$, $[3^6 : 3^3.4^2]_2$, $[3^6 : 3^2.4.3.4]$, $[3^3.4^2 : 3^2.4.3.4]_1$, $[3^3.4^2 : 3^2.4.3.4]_2$, $[3^3.4^2 : 4^4]_1$ and $[3^3.4^2 : 4^4]_2$. Their method of classification becomes more complex as the number of vertices increases. To overcome (avoid) this difficulty, here, we give a construction to classify doubly semi-equivelar maps on torus corresponding to the 2-uniform tilings for arbitrary number of vertices. We describe this construction corresponding to the above seven types tilings. A study of doubly semi-equivelar maps for Klein bottle is carried out in [12].

We proceed the article as follows. In Section 2, we start with the preliminaries. Here, we describe semi-equivelar and doubly semi-equivelar maps. We conclude this section by presenting the aim of this article explicitly. In Section 3, we give a constriction to classify and enumerate doubly semi-equivelar maps for the seven types 2-uniform tilings. Moreover, we enumerate such maps for different number of vertices.

2 Definitions and notations

Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$ respectively. The notation u-v denotes an edge joining $u, v \in V(G)$. Let $Q = C_k(v_1, \ldots, v_k)$ denote a cycle v_1-v_2- $\cdots$ -v_k-v_1 of length k. A cycle Q is called contractible it is the boundary of a 2-disk D_p, $p \geq 3$, otherwise called non-contractible. Let $P = P(u_1, \ldots, u_n)$ denote a path u_1-u_2- $\cdots$ -u_n. The vertices u_i, $2 \leq i \leq n-1$, are called inner vertices and u_1, u_2 are called boundary vertices of the path. A path P_1 is called an extension of another path P_2 if P_2 is a proper subgraph of P_1. The notations $G_1 \cup G_2$ and $G_1 \cap G_2$ denote the usual union and intersection of two graphs G_1 and G_2. These definitions also can be seen in [2].

Let F be a surface (closed) and G be a connected, simple graph. An embedding of G into F is called a map M if the closure of each components of $F \setminus G$ is a p-gonal 2-disk D_p, $p \geq 3$. Thus there are precisely three components in a map, namely the disks, also called faces, vertices and edges. The vertices and edges of M are nothing but the vertices and edges of the underlying graph G of M. A map is called a polyhedral map if the non-empty intersection of any two distinct faces is either a vertex or an edge, see [4]. Throughout this article, by a map we mean a polyhedral map. Two maps M_1 and M_2, with vertex sets $V(M_1)$ and $V(M_2)$ respectively, are isomorphic, denoted as $M_1 \cong M_2$, if there is a bijective map $f : V(M_1) \to V(M_2)$ which preserves the incidence of edges and incidence of faces in the maps. The face-sequence of a vertex v, denoted as $f_{seq}(v)$, in a map M is a finite sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$ such that n_1 numbers of D_{p_1}, n_2 numbers of D_{p_2}, $\ldots$, n_k numbers of D_{p_k} incident at v in the given cyclic order. A map M is called semi-equivelar of type $[p_1^{n_1}. \ldots .p_k^{n_k}]$ if the face-sequence of each vertex is same, i.e., $(p_1^{n_1}, \ldots, p_k^{n_k})$, see [10].

Let v be a vertex in a map M such that $D_{p_1}, \ldots, D_{p_k}$ be consecutive faces around v, i.e., the face-cycle at v is $(D_{p_1}, \ldots, D_{p_k})$. Let C_{p_i} be the boundary of D_{p_i}, for $1 \leq i \leq k$. Then the link of v, denoted as lk(v), is a cycle in the underlying graph of M containing all the vertices of C_{p_i}'s except v and all the edges of C_{p_i}'s except which has one end vertex v. If v is a vertex with lk$(v) = C_k(v_1, \ldots, v_k)$ then the face-sequence of lk(v) is a cyclically ordered sequence $(f_{seq}(v_1), \ldots, f_{seq}(v_k))$.

Let v be a vertex with the face sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$. The combinatorial curvature of v, denoted by $\phi(v)$, is defined as $\phi(v) = 1 - (\sum_{i=1}^{k} n_i)/2 + (\sum_{i=1}^{k} n_i/p_i))$.

We now generalize the definition of semi-equivelar map for the maps having two distinct face-sequences under some restrictions and call such map as doubly semi-equivelar map.

Suppose, M is a map with two distinct face-sequences f_1 and f_2. Then M is called a doubly semi-equivelar map [11], in short DSEM, if (i) $\phi(v)$ has same sign (i.e., either negative, 0 or positive) for all $v \in V(M)$, and (ii) the vertices of same face-sequence also have links of same face-sequence up to a cyclic permutation. We denote the M of type $[f_1^{(f_{11},\ldots,f_{1r_1})} : f_2^{(f_{21},\ldots,f_{2r_2})}]$, where $f_{1i}, f_{2j} \in \{f_1, f_2\}$, for $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$, if vertices of the face-sequence f_1 have links of face-sequence $(f_{11}, \ldots, f_{1r_1})$ and vertices of the face-sequence f_2 have links of face-sequence $(f_{21}, \ldots, f_{2r_2})$.

The present work is motivated by an attempt to enumerate doubly semi-equivelar maps corresponding to the 2-uniform tilings which have p-gon tiles with $p \leq 4$, that is, the tilings: $[3^6 : 3^3.4^2]_2$, $[3^6 : 3^2.4.3.4]$, $[3^3.4^2 : 3^2.4.3.4]_1$, $[3^3.4^2 : 3^2.4.3.4]_2$, $[3^3.4^2 : 4^4]_1$, $[3^3.4^2 : 4^4]_2$, $[3^6 : 3^4.6]_1$. Their respective DSEM types are given in Table 2 of [11]. For simplicity, the types of these DSEMs are denoted by the same notations as used for the respective tilings. We show:

Theorem 2.1 *The doubly semi-equivelar maps with n vertices of type X, where $X = [3^6 : 3^3.4^2]_1$, $[3^6 : 3^3.4^2]_2$, $[3^6 : 3^2.4.3.4]$, $[3^3.4^2 : 3^2.4.3.4]_1$, $[3^3.4^2 : 3^2.4.3.4]_2$, $[3^3.4^2 : 4^4]_1$ or $[3^3.4^2 : 4^4]_2$, can be classified on torus up to the isomorphism.*

Moreover we enumerate the above types DSEMs for few vertices, see Table 3.1-3.7.

3 Classification of doubly semi-equivelar maps on torus

Altshuler [1] has given an algorithmic approach to enumerate semi-equivelar maps on torus for the types $[3^6]$, $[4^4]$ and $[6^3]$. Maity and Upadhyay [14] generalized their approach for the remaining types of semi-equivelar maps on torus. The present construction to enumerate doubly semi-equivelar maps is motivated by the theory of semi-equivelar maps given in [14]. Throughout this section, by a map we mean a map on torus. Let u be a vertex with face-sequence (3^6), $(3^3, 4^2)$, $(3^2, 4, 3, 4)$ or (4^4). We denote their respective link by $\mathrm{lk}(u) = C_6(u_1, u_2, u_3, u_4, u_5, u_6), \mathrm{lk}(u) = C_7(\boldsymbol{u_1}, u_2, \boldsymbol{u_3}, u_4, u_5, u_6, u_7), \mathrm{lk}(u) = C_7(u_1, u_2, u_3, \boldsymbol{u_4}, u_5, u_6, \boldsymbol{u_7})$ or $\mathrm{lk}(u) = C_8(u_1, \boldsymbol{u_2}, u_3, \boldsymbol{u_4}, u_5, \boldsymbol{u_6}, u_7, \boldsymbol{u_8})$ (labeling of vertices in the links are considered anti-clockwise). The bold appearance of some u_i means u is not adjacent with u_i by an edge. We use these notations frequently in this section.

3.1 DSEMs of type $[3^6 : 3^3.4^2]_1$

Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$ with the vertex set $V(M)$. Let $V_{(3^6)}$ and $V_{(3^3,4^2)}$ denote the vertex sets with face-sequences (3^6) and $(3^3, 4^2)$ respectively. Clearly $V(M) = V_{(3^6)} \cup V_{(3^3,4^2)}$. It is easy to see that the number of triangular faces in M is $4|V_{(3^6)}|$ or $2|V_{(3^3,4^2)}|$, where $|V_{(3^6)}|$ and $|V_{(3^3,4^2)}|$ denote the respective cardinality of sets $V_{(3^6)}$ and $V_{(3^3,4^2)}$. Thus for existence of M, we have $4|V_{(3^6)}| = 2|V_{(3^3,4^2)}|$, that is, $2|V_{(3^6)}| = |V_{(3^3,4^2)}|$. Now consider the following three paths P_1, P_2 and P_3 in M as follows.

Definition 3.1.1 A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M is of type A_1 if all the triangles incident with an inner vertex y_i lie on one side and all the quadrangles incident with y_i lie on the other side of the subpath $P'(y_{i-1}, y_i, y_{i+1})$ or every vertex of the path have face-sequence (3^6), see Figure 3.1.1. If a boundary vertex of P_1 is y_j then there is an extended path say P_e of P_1 s.t. y_j is an inner vertex of P_e.

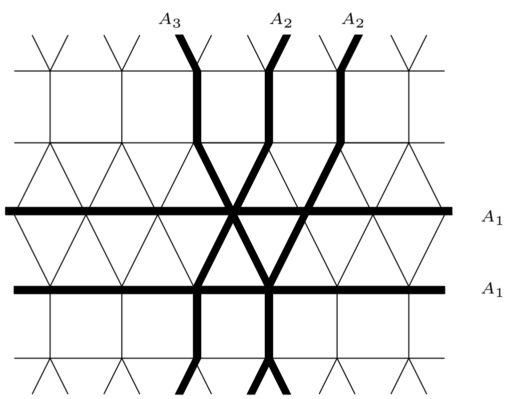

Figure 3.1.1: Paths of types A_1, A_2, A_3

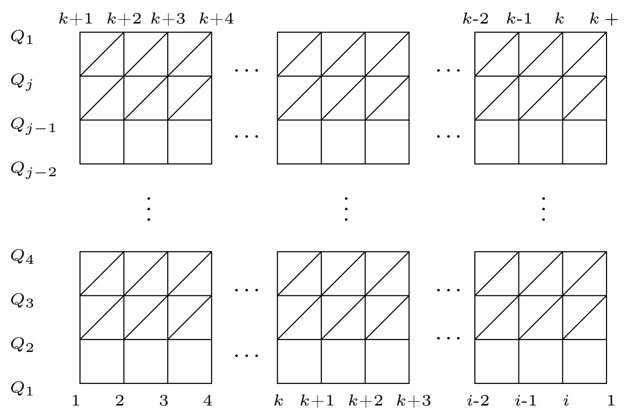

Figure 3.1.2: $M(i, j, k)$

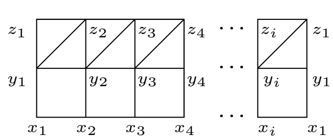

Figure 3.1.3: Cylinder-I

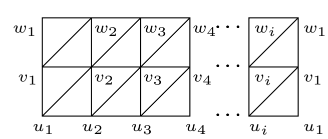

Figure 3.1.4: Cylinder-II

Definition 3.1.2 Consider a path $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M such that z_i, z_{i+1} are inner vertices of P_2 or an extended path of P_2. We say that P_2 is of type A_2, see Figure 3.1.1, if either of the following two conditions occur for each vertex of the path.

1. if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, z_{i+1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_6(z_i, o, r, z_{i+2}, s, p)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{t}, z_{i+3}, \boldsymbol{u}, s, z_{i+1}, r, v)$, and if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, z_{i-1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{o}, z_i, \boldsymbol{q}, m, z_{i+2}, r, n)$, $\mathrm{lk}(z_{i+2}) = C_6(z_{i+1}, m, s, z_{i+3}, t, r)$.

2. if $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{q}, z_{i+2}, \boldsymbol{r}, o, n, z_i, n, s)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{o}, z_{i+1}, \boldsymbol{s}, q, z_{i+3}, t, r)$, and if $\mathrm{lk}(z_i) = C_6(z_{i+1}, m, n, z_{i-1}, o, p)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{q}, z_{i+2}, \boldsymbol{r}, m, z_i, p, s)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{s}, q, z_{i+3}, t, r)$.

Definition 3.1.3 Consider a path $P_3 = P(\ldots, w_{i-1}, w_i, w_{i+1}, \ldots)$ in M such that w_i, w_{i+1} are inner vertices of P_3 or an extended path of P_3. We say that P_3 is of type A_3, see Figure 3.1.1, if either of the following two conditions occur for each vertex of the path

1. if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i-1}, \boldsymbol{n}, o, p, w_{i+1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_6(q, w_i, p, r, w_{i+2}, s)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{t}, w_{i+3}, \boldsymbol{u}, v, s, w_{i+1},)$, and if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, p, w_{i-1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{o}, w_i, \boldsymbol{q}, m, r, w_{i+2}, n)$, $\mathrm{lk}(w_{i+2}) = C_6(w_{i+1}, r, s, w_{i+3}, t, n)$.

2. $\mathrm{lk}(w_i) = C_6(m, w_{i-1}, n, o, w_{i+1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{q}, w_{i+2}, \boldsymbol{r}, s, p, w_i, o)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+1}, o, q, \boldsymbol{t}, w_{i+3}, r)$, and if $\mathrm{lk}(w_i) = C_6(m, w_{i+1}, n, o, w_{i-1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{q}, w_{i+2}, \boldsymbol{r}, s, n, w_i, m)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+1}, \boldsymbol{m}, q, t, w_{i+3}, r)$.

Since M is map on finite vertices, if $P(v_1, v_2, \ldots, v_l)$ is a maximal path (a path of maximum length) of type A_α, $\alpha \in \{1, 2, 3\}$ in M then the path gives a cycle $Q = C_l(v_1, v_2, \ldots, v_l)$ (see Lemma 4.1 in [14]). In other words:

Lemma 3.1.1 *Let $P(v_1, v_2, \ldots, v_l)$ be a maximal path in M of the type A_α, $\alpha \in \{1, 2, 3\}$. Then there is an edge v_l-v_1 in the underlying graph of M such that $P(v_1, v_2, \ldots, v_l) \cup \{v_l\text{-}v_1\}$ is a cycle $Q = C_l(v_1, v_2, \ldots, v_l)$.*

In the above lemma, if the Q is obtained by a maximal path $P(v_1, v_2, \ldots, v_l)$ of type A_1 (resp. A_2 or A_3), we call Q of type A_1 (resp. A_2 or A_3). Let Q_1, Q_2 be two cycles of same type in M such that $E(Q_1) \cap E(Q_2) \neq \phi$, where $E(Q_i)$ denotes the edge set of Q_i for $1 \leq i \leq 2$. Then by the argument given for Lemma 4.2 in [14], we see easily that they are identical. That is:

Lemma 3.1.2 *If Q_1, Q_2 are two cycles of type A_α, for a fixed $\alpha \in \{1, 2, 3\}$, in M. Then, $Q_1 = Q_2$ if $E(Q_1) \cap E(Q_2) \neq \phi$.*

For a cycle Q of type A_α, where $\alpha \in \{1, 2, 3\}$, we show the following.

Lemma 3.1.3 *If Q is a cycle of type A_α, for $\alpha \in \{1, 2, 3\}$, in M then Q is non-contractible.*

Proof. We prove it by contradiction. Suppose Q is a contractible cycle of type A_1. Then Q is the boundary of a 2-disk D_Q. Let $|V|$, $|E|$ and $|F|$ denote the number of vertices, edges and faces of D_Q respectively. Suppose there are $k = k_1 + k_2$ inner vertices and l boundary vertices in D_Q. Here k_1 and k_2 denote the number of inner vertices with face-sequences (3^6) and $(3^3, 4^2)$ respectively. By Definition 3.1.1, we have two cases for $|V|$, $|E|$ and $|F|$ in D_Q. In case when quadrangles are incident with Q then $|V| = l + k_1 + k_2$, $|E| = 3l/2 + 6k_1/2 + 5k_2/2$ and $|F| = 2l/4 + 6k_1/3 + 3k_2/3 + 2k_2/4$, on the other hand when triangles are incident with Q then $|V| = l + k_1 + k_2$, $|E| = 4l/2 + 6k_1/2 + 5k_2/2$ and $|F| = 3l/3 + 6k_1/3 + 2k_2/4 + 3k_2/3$. But for both the cases, we get $|V| - |E| - |F| = 0$. A contradiction, as the Euler characteristic of a 2-disk is 1. Thus Q is non-contractible.

Let Q be a contractible cycle of type A_2. Let D_Q be a 2-disk with the boundary Q. Suppose there are $k = k_1 + k_2$ inner vertices and $l = l_1 + l_2 + l_3$ boundary vertices in D_Q, where k_1 and k_2 denote the number of inner vertices with face-sequences (3^6) and $(3^3, 4^2)$ respectively and l_1, l_2, l_3 denote the number of boundary vertices of type $(3, 4)$, (3^3) and $(3^2, 4)$ respectively (by a boundary vertex v of type $(3^r, 4^s)$ means D_Q contains r triangles and s quadrangles consecutively incident with v). Then $|V| = l_1 + l_2 + l_3 + k_1 + k_2$, $|E| = 3l_1/2 + 4l_2/2 + 4l_3/2 + 6k_1/2 + 5k_2/2$ and $|F| = l_1/4 + l_1/3 + 3l_2/3 + 2l_3/3 + l_3/4 + 6k_1/3 + 3k_2/3 + 2k_2/4$. Note that $l_1 = l_2 = l_3$, we get $|V| - |E| - |F| = 0$, a contradiction. Thus Q is non-contractible.

Similarly we see, if Q is a cycle of type A_3 then it is also non-contractible. Thus the proof. $\square$

Let Q be a cycle of type A_α with the vertex set $V(Q)$, for a fixed $\alpha \in \{1, 2, 3\}$ in M. Let S denote the set of all the faces incident at v for all $v \in V(Q)$. Then the geometric carrier S_Q (union of all the faces in S) is a cylinder, as Q is non-contractible (for example see Figure 3.1.3 and 3.1.4). By Lemma 3.1.2, it has two disjoint or identical boundary cycles, say Q_1 and Q_2. Let $\partial S_Q = \{Q_1, Q_2\}$, where ∂S_Q denote the boundary of S_Q. Then we show:

Lemma 3.1.4 *If Q be a cycle of type A_α, $\alpha \in \{1, 2, 3\}$, such that $\partial S_Q = \{Q_1, Q_2\}$. Then (i) Q, Q_1 and Q_2 are of same type, (ii) length(Q) = length(Q_1) = length(Q_2).*

Proof. Let Q be a cycle of type A_1 such that S_Q is a cylinder with $\partial S_Q = \{Q_1, Q_2\}$. Consider the faces which are incident with both Q and Q_n for a fixed $n \in \{1, 2\}$. Without loss of generality let $Q_n = Q_1$. Now depending on Q we have two cases: (i) if the faces incident with both Q and Q_1 are quadrangles then the faces lying on the other side of Q_1 must be triangles and (ii) if the faces incident with both Q and Q_1 are triangles then the faces on the other side of Q_1 must be either triangles or quadrangles. Following Definition 3.1.1, we see for both the cases, Q_1 is of type A_1 and hence Q_2 is also of type A_1.

Now let $Q = C_l(v_1, \ldots, v_l)$, $Q_1 = C_{l_1}(u_1, \ldots, u_{l_1})$ and $Q_2 = C_{l_2}(w_1, \ldots, w_{l_2})$. We show that $l = l_1 = l_2$. Suppose $l \neq l_1 \neq l_2$. Without loss of generality let $l < l_1 < l_2$. By Definition 3.1.1, the face-sequence of $v_1, v_2, \ldots, v_{l-1}, v_l$ will be same through out the cycle. Since $l < l_1$, so, the lk(v_l) contains the vertices u_l, u_{l+i}, and w_l for some $i > 0$. This shows that the face-sequences of v_l and v_{l-1} are not same. A contradiction. Therefore $l = l_1 = l_2$.

Proceeding similarly, we get the above result for a cycles of type A_2 and A_3. $\square$

Let Q_1 and Q_2 be two same type cycles in a DSEM M on the torus. We say that cycles Q_1 and Q_2 are homologous if there is a cylinder whose boundary is $\{Q_1, Q_2\}$, see [14]. Thus in Lemma 3.1.4, the cycles Q, Q_1 and Q_2 are homologous to each other.

Now we give the notion of a planar representation, denoted as $M(i, j, k)$ representation, for a DSEM M (as defined for semi-equivelar maps in [14]). This is obtained by cutting M along any two non-homologous cycles.

An $M(i, j, k)$ representation: Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$ with the vertex set $V(M)$. Let $u \in V(M)$ and Q_α be cycles of type A_α through u, where $\alpha \in \{1, 2, 3\}$. Let $Q_1 = C_i(u_1, u_2, \ldots, u_i)$. We cut M first along the cycle Q_1. We get a cylinder, say R_1, bounded by identical cycle Q_1. We say that a cycle is horizontal if it is Q_1 or homologous to Q_1. Then we say that a cycle is vertical if it is Q_α or homologous to Q_α for $\alpha \in \{2, 3\}$. In R_1, starting from the vertex u, make another cut along the path $P \subset Q_3$, until it reaches Q_1 again for the first time. As a result, we unfold the torus into a planer representation, say R_2.

Claim. The representation R_2 is connected.

Since Q_1 is non-contractible cycle, R_1 is connected. Suppose that R_2 is disconnected. This means there is a 2-disk D_Q with boundary cycle $Q = P_1 \cup P_1' = P(a_m, \ldots, a_n) \cup P(b_s, \ldots, b_t)$ where $P_1 \subset Q_3$, $P_1' \subset Q_1$, $a_n = b_s$ and $a_m = b_t$. Let $\triangle$ and $\square$ denote triangular and quadrangular faces respectively. Now in the first case when quadrangular faces are incident with P_1' in D_Q and $\square_m$, $\triangle_{m,1}$, $\triangle_{m,2}$, $\triangle_{m,3}$, $\triangle_{m,4}$, $\square_{m+1}$, $\triangle_{m+1,1}$, $\triangle_{m+1,2}$, $\triangle_{m+1,3}$, $\triangle_{m+1,4}, \ldots, \triangle_{n-1,1}$, $\triangle_{n-1,2}$, $\triangle_{n-1,3}$, $\triangle_{n-1,4}$, $\square_n$ are incident with P_1 in D_Q then as in Lemma 3.1.3, we calculate $|V|$, $|E|$ and $|F|$ of D_Q and we get $|V| - |E| + |F| = 0$. On the other hand, if triangular faces are incident with P_1' in D_Q then again calculating $|V|$, $|E|$ and $|F|$ of D_Q, we get $|V| - |E| + |F| = 0$. This is not possible as the Euler characteristic of the 2-disk D_Q is 1. So, R_2 is connected.

Without loss of generality, suppose that the quadrangular faces are incident on Q_1. In R_2, let there are j cycles which are homologous to Q_1 along the path $P \subset Q_3$. Since length$(Q_1) = i$ and the number of horizontal cycles along P is j, as shown in Figure 3.1.2. So, we denote R_2 by (i, j) representation. Now to obtain M from its (i, j) representation one has to go in reverse way, i.e., identify both the vertical sides and horizontal sides along the vertices, but observe that identification of the horizontal sides may need some shifting, as shown in Figure 3.1.2. Let k be the shifting, i.e., u_{k+1} be the first vertex in the upper horizontal cycle. This gives a planar representation of M (corresponding to the (i, j) representation) also called $M(i, j, k)$ representation of M.

Lemma 3.1.5 *In $M(i, j, k)$, A_1 type cycles have unique length and A_2 type (or A_3 type) cycles have at most two different lengths.*

Proof. Consider an $M(i, j, k)$ representation of a DSEM M of type $[3^6 : 3^3.4^2]_1$ with the vertex set $V(M)$. Let Q_1 be an A_1 type cycle. Consider a cylinder S_{Q_1} which is defined by Q_1. Let $\partial S_{Q_1} = \{Q_0, Q_2\}$. By Lemma 3.1.4, the cycles Q_0, Q_1 and Q_2 are homologous and length(Q_0) = length(Q_1) = length(Q_2). Now consider the cycle Q_2 and repeat the above procedure. In this process, let Q_m indicate a cycle at m^{th} step such that $\partial S_{Q_m} = \{Q_{m-1}, Q_{m+1}\}$ and length(Q_{m-1}) = length(Q_m) = length(Q_{m+1}). Since $V(M)$ is finite, this process terminates after finite number of steps. By the construction of $M(i, j, k)$, shown in Figure 3.1.2, the process stops after $j+1$ number of steps, i.e., Q_0 appears again. Thus, the cycles Q_r, Q_s are homologous for every $1 \le r, s \le j$ and $\bigcup_{m=1}^{j} V(Q_m) = V(M)$. Note that in $M(i, j, k)$ there is only one cycle of type A_1 through any vertex. As a result, $Q_1, Q_2, \ldots, Q_j$ are the only type A_1 cycles in M. Since these cycles are homologous and length(Q_1) = length(Q_r) for all $1 \le r \le j$. Therefore, M has A_1 type cycles with unique length.

Let Q_1', Q_1'' be the cycles of type A_2 and A_3 respectively. Now repeating the above process for the cycles Q_1' and Q_1'', we see that all the A_2 type cycles have same length say l_1 and all the A_3 type cycles have same length say l_2. Observe that Q_1' and Q_1'' define same type cycles as these are mirror image of each other. So, the map M contains the cycles of type A_2 (or type A_3) of lengths l_1 and l_2. Therefore, M has A_2 type (or A_3 type) cycles with at most two different lengths. $\Box$

Now we define a cycle of new type (other than A_1, A_2 and A_3) as follows: Suppose that a DSEM M of type $[3^6 : 3^3.4^2]_1$ has an $M(i, j, k)$ representation. Let $Q_{lh} = C_i(x_1, x_2, \ldots, x_i)$ and $Q_{uh} = C_i(x_{k+1}, x_{k+2}, \ldots, x_k)$ be the lower and upper horizontal cycles of type A_1 in the representation respectively. Define two paths $P_1 = P(x_{k+1}, y_1, y_2, \ldots, y_\alpha, x_{k_1})$ of type A_2 and $P_1' = P(x_{k+1}, z_1, z_2, \ldots, z_\beta, x_{k_2})$ of type A_3 through x_{k+1} in the representation, where $x_{k_1}, x_{k_2} \in V(Q_{uh})$. Note that, the paths P_1 and P_1' are not parts of horizontal cycles. Consider the paths $P_2 = P(x_{k_1}, \ldots, x_{k+1})$ and $P_2' = P(x_{k_2}, \ldots, x_{k+1})$ in Q_{uh} such that $P_2 \cup P_2' \subseteq Q_{uh}$. Let $Q_{4,1} = P_1 \cup P_2 = C_\gamma(x_{k+1}, y_1, y_2, \ldots, y_\alpha, x_{k_1}, \ldots, x_{k+1})$ and $Q_{4,2} = P_1' \cup P_2' = C_{\gamma'}(x_{k+1}, z_1, z_2, \ldots, z_\beta, x_{k_2}, \ldots, x_{k+1})$, where γ and γ' are lengths of $Q_{4,1}$ and $Q_{4,2}$ respectively. Define a cycle Q_4 of new type, say A_4, as:

$$Q_4 = \begin{cases} Q_{4,1}, & \text{if } \gamma(Q_{4,1}) \le \gamma'(Q_{4,2}) \\ Q_{4,2}, & \text{if } \gamma(Q_{4,1}) > \gamma'(Q_{4,2}) \end{cases} \tag{1}$$

From (1), clearly the length(Q_4) = min{length(P_1) + length(P_2), length (P_1') + length(P_2')} = $\min\{k + j, (i - k - 2j/3)(mod\, i) + j\}$. Here we denote $(i - k - 2j/3)$ for $(i - k - 2j/3)(mod\, i)$.

From the notion of $M(i, j, k)$ representation, we prove the following lemma.

Lemma 3.1.6 *Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$. Then M admits an $M(i, j, k)$ representation if and only if the following holds: (i) $i \ge 3$ and $j = 3m$, where $m \in \mathbb{N}$, (ii) $ij \ge 9$, (iii) $0 \le k \le i-1$.*

Proof. Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$ with n vertices. By definition $M(i, j, k)$ of M has j number of A_1 type disjoint horizontal cycles of length i. Since all the vertices of M lie in these cycles, the number of vertices in M is $n = ij$. Clearly if $i \leq 2$, M is not a map. So $i \geq 3$. If $j = 1$ then M is not a map and if $j = 2$ then M has no vertices with face-sequence (3^6). If $j = 3m+1$ or $3m+2$, where $m \in \mathbb{N}$, then $2|V_{(3^6)}| \neq |V_{(3^3,4^2)}|$. So $j = 3m$ for $m \in \mathbb{N}$. Thus $n = ij \geq 9$. Since the length of the horizontal cycle is i, we get $0 \leq k \leq i-1$. Converse is obvious by Figure 3.1.2. This completes the proof. $\square$

Let M_t, $t = 1, 2$, be DSEMs of type $[3^6 : 3^3.4^2]_1$ on n_t number of vertices with $n_1 = n_2$. Let $M_t(i_t, j_t, k_t)$ be representation of M_t. Let $Q_{t,\alpha}$ be cycles of type A_α and $l_{t,\alpha}$ = length of the cycle of type A_α, $\alpha = 1, 2, 3, 4$, in $M_t(i_t, j_t, k_t)$. We say that $M_t(i_t, j_t, k_t)$ has cycle-type $(l_{t,1}, l_{t,2}, l_{t,3}, l_{t,4})$ if $l_{t,2} \leq l_{t,3}$ or $(l_{t,1}, l_{t,3}, l_{t,2}, l_{t,4})$ if $l_{t,3} < l_{t,2}$. Now, we show the following.

Lemma 3.1.7 *The DSEMs $M_1 \cong M_2$ iff they have same cycle-type.*

Proof. Let M_1 and M_2 be two DSEMs. Suppose the maps have same cycle-type. Then $l_{1,1} = l_{2,1}$, $\{l_{1,2}, l_{1,3}\} = \{l_{2,2}, l_{2,3}\}$ and $l_{1,4} = l_{2,4}$. To show that $M_1 \cong M_2$, it is enough to show that $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Claim. $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

By the definition, $M_t(i_t, j_t, k_t)$ has j_t horizontal cycles of type A_1, say $Q_0 = C_{i_1}(w_{0,0}, w_{0,1}, \ldots, w_{0,i_1-1}), Q_1 = C_{i_1}(w_{1,0}, w_{1,1}, \ldots, w_{1,i_1-1}), \ldots, Q_{j_1-1} = C_{i_1}(w_{j_1-1,0}, w_{j_1-1,1}, \ldots, w_{j_1-1,i_1-1})$ in $M_1(i_1, j_1, k_1)$ and $Q'_0 = C_{i_2}(x_{0,0}, x_{0,1}, \ldots, x_{0,i_2-1}), Q'_1 = C_{i_2}(x_{1,0}, x_{1,1}, \ldots, x_{1,i_2-1}), \ldots, Q'_{j_2-1} = C_{i_2}(x_{j_2-1,0}, x_{j_2-1,1}, \ldots, x_{j_2-1,i_2-1})$ in $M_2(i_2, j_2, k_2)$. Then we have the following cases.

Case 1: If $(i_1, j_1, k_1) = (i_2, j_2, k_2)$ then $i_1 = i_2, j_1 = j_2, k_1 = k_2$. Define an isomorphism $f : V(M_1(i_1, j_1, k_1)) \to V(M_2(i_2, j_2, k_2))$ such that $f(w_{u,v}) = x_{u,v}$ for $0 \leq u \leq j_1 - 1$ and $0 \leq v \leq i_1 - 1$. So, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 2: If $i_1 \neq i_2$ then it contradicts the fact that $l_{1,1} = l_{2,1}$. Thus $i_1 = i_2$.

Case 3: If $j_1 \neq j_2$ then $n_1 = i_1 j_1 \neq i_2 j_2 = n_2$ as $i_1 = i_2$. A contradiction, as $n_1 = n_2$. So, $j_1 = j_2$.

Case 4: If $k_1 \neq k_2$. Since $l_{1,4} = l_{2,4}$, length$(Q_{1,4})$ = length$(Q_{2,4})$. This means $\min\{k_1 + j_1, i_1 - k_1 + j_1/3\} = \min\{k_2 + j_2, i_2 - k_2 + j_2/3\}$. Since $i_1 = i_2$, $j_1 = j_2$ and $k_1 \neq k_2$, we get $k_1 + j_1 \neq k_2 + j_2$ and $i_1 - k_1 + j_1/3 \neq i_2 - k_2 + j_2/3$. This gives that $k_1 + j_1 = i_2 - k_2 + j_2/3 = i_1 - k_2 + j_1/3$, as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1 - k_1 - 2j_1/3$. We identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(x_{0,0}, x_{1,0}, \ldots, x_{j_2-1,0}, x_{0,k_2})$ and then cut along the path $P(x_{0,0}, x_{1,0}, x_{2,1}, x_{3,2}, \ldots, x_{j_2-1,2j_2/3-1}, x_{0,k_2+2j_2/3})$ of type A_2 through vertex $x_{0,0}$. This gives another representation of M_2, say R, with a map $f' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that $f'(x_{u,v}) = x_{u,(i_2-v+\lfloor 2u/3 \rfloor)(mod\, i_2)}$ for $0 \leq u \leq j_2 - 1$ and $0 \leq v \leq i_2 - 1$. In R the lower and upper horizontal cycles are $Q' = C_{i_2}(x_{0,0}, x_{0,i_2-1}, x_{0,i_2-2}, \ldots, x_{0,1})$ and $Q'' = C_{i_2}(x_{0,k_2+2j_2/3}, x_{0,k_2+2j_2/3-1}, \ldots, x_{0,k_2+2j_2/3+1})$ respectively. The path $P(x_{0,0}, x_{0,i_2-1}, x_{0,i_2-2}, \ldots, x_{0,k_2+2j_2/3})$ in Q' has length $i_2 - k_2 - 2j_2/3$. Clearly R has j_2 number of horizontal cycles of length i_2. So, $R = M_2(i_2, j_2, i_2 - k_2 - 2j_2/3)$. Note that $i_2 - k_2 - 2j_2/3 = i_2 - (i_1 - k_1 - 2j_1/3) - 2j_2/3 = k_1$, as $i_1 = i_2$ and $j_1 = j_2$. So, $M_2(i_2, j_2, i_2 - k_2 - 2j_2/3) = M_1(i_1, j_1, k_1)$. Therefore by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$. So, by Cases 1-4, the claim follows. Hence, $M_1 \cong M_2$.

Conversely, let $M_1 \cong M_2$ by an isomorphism f. Let $Q_{1,\alpha}$ and $Q_{2,\alpha}$ be cycles of type A_α for $\alpha = 1, 2, 3, 4$ in M_1 and M_2 respectively. Let $f : V(M_1) \to V(M_2)$ be such that $f(Q_{1,\alpha}) = Q_{2,\alpha}$. Since f is an isomorphism, length$(Q_{1,\alpha})$ = length$(f(Q_{1,\alpha}))$ = length$(Q_{2,\alpha})$. Hence, M_1 and M_2 have the same cycle-type. This completes the proof. $\square$

By Lemma 3.1.6 and 3.1.7, one can compute and classify DSEMs of type $[3^6 : 3^3.4^2]$ for different number of vertices $|V(M)|$. A tabular list of the DSEMs for first four admissible values of $|V(M)|$, i.e., $|V(M)| = 9, 12, 15, 18$ is given in Table 3.1. For $|V(M)| = 12$, the computation is illustrated explicitly in the following example.

Example 3.1.1 Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$ with n vertices. Then M has an $M(i, j, k)$ representation, with $n = ij \geq 9$, $j = 3m$, where $m \in \mathbb{N}$ and $0 \leq k \leq i-1$. If $n = 12$ and $j = 3$, we have, $i = 4$ and $k = 0, 1, 2, 3$ by Lemma 3.1.6. So, $M(i, j, k) = M(4, 3, 0), M(4, 3, 1), M(4, 3, 2)$ and $M(4, 3, 3)$, see Figures 3.1.5, 3.1.6, 3.1.7, 3.1.8, respectively. In $M(4, 3, 0)$, $Q_{1,1} = C_4(v_1, v_2, v_3, v_4)$ is A_1 type cycle, $Q_{1,2} = C_6(v_1, v_5, u_2, v_3, v_7, u_4)$ and $Q_{1,3} = C_3(v_1, v_5, u_1)$ are two A_2 type cycles and $Q_{1,4} = C_3(v_1, v_5, u_1)$ is A_4 type cycle. In $M(4, 3, 1)$, $Q_{2,1} = C_4(w_1, w_2, w_3, w_4)$ is A_1 type cycle, $Q_{2,2} = C_{12}(w_1, w_5, u_2, w_4, w_8, u_1, w_3, w_7, u_4, w_2, w_6, u_3)$ and $Q_{2,3} = C_{12}(w_1, w_5, u_1, w_2, w_6, u_2, w_3, w_7, u_3, w_4, w_8, u_4)$ are two A_2 type cycle and $Q_{2,4} = C_4(w_2, w_6, u_3, w_1)$ is A_4 type cycle. In $M(4, 3, 2)$, $Q_{3,1} = C_4(x_1, x_2, x_3, x_4)$ is a cycle of type A_1, $Q_{3,2} = C_3(x_1, x_5, u_2)$ and $Q_{3,3} = C_6(x_1, x_5, u_1, x_3, x_7, u_3)$ are two A_2 type cycles and $Q_{3,4} = C_3(x_3, x_7, u_4)$ is A_4 type cycle. In $M(4, 3, 3)$, $Q_{4,1} = C_4(z_1, z_2, z_3, z_4)$ is A_1 type cycle, $Q_{4,2} = C_{12}(z_1, z_5, u_2, z_2, z_6, u_3, z_3, z_7, u_4, z_4, z_8, u_1)$ and $Q_{4,3} = C_{12}(z_1, z_5, u_1, z_4, z_8, u_4, z_3, z_7, u_3, z_2, z_6, u_2)$ are two A_2 type cycles and $Q_{4,4} = C_6(z_4, z_8, u_1, z_1, z_2, z_3)$ is A_4 type cycle.

By Lemma 3.1.5, M has A_1 type cycles with unique length and A_2 type cycles with at most two different lengths. Since length$(Q_{1,4}) \neq$ length$(Q_{r,4})$ for $r = 2, 4$, $M(4, 3, 0) \ncong M(4, 3, 1), M(4, 3, 3)$. Also, $M(4, 3, 1) \ncong M(4, 3, 2)$ as length$(Q_{2,4}) \neq$ length$(Q_{3,4})$ and $M(4, 3, 2) \ncong M(4, 3, 3)$ as length $(Q_{3,4}) \neq$ length$(Q_{4,4})$. Observe that, length$(Q_{1,1})$ = length$(Q_{3,1})$, {length$(Q_{1,2})$, length$(Q_{1,3})$} = {length$(Q_{3,2})$, length$(Q_{3,3})$} and length$(Q_{1,4})$ = length$(Q_{4,4})$. Now identifying $M(4, 3, 0)$ along the vertical boundary and cutting along the path $P(v_1, v_5, u_2, v_3)$ leads to Figure 3.1.9, i.e., $M(4, 3, 2)$. So, by Lemma 3.1.7, $M(4, 3, 0) \cong M(4, 3, 2)$. Thus, there are three DSEMs, up to isomorphism, of type $[3^6 : 3^3.4^2]_1$ with 12 vertices on the torus. These are $M(4, 3, 0)$, $M(4, 3, 1)$, $M(4, 3, 3)$.

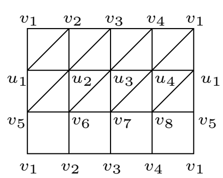

Figure 3.1.5: $M(4, 3, 0)$

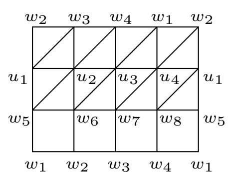

Figure 3.1.6: $M(4, 3, 1)$

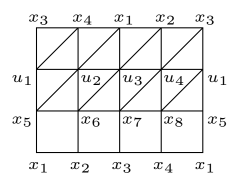

Figure 3.1.7: $M(4, 3, 2)$

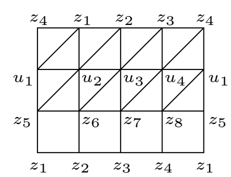

Figure 3.1.8: $(4, 3, 3)$

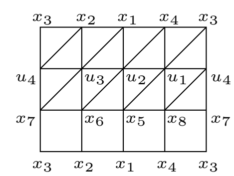

Figure 3.1.9: $M(4, 3, 0)$

Table 3.1 : DSEMs of type $[3^6 : 3^3.4^2]_1$ on the torus for $|V(M)| \leq 18$

$\|V(M)\|$	Isomorphic classes	Length of cycles	No of maps
9	$M(3,3,0)$, $M(3,3,1)$	$(3, \{3, 9\}, 3)$	2
	$M(3,3,2)$	$(3, \{9, 9\}, 5)$	
12	$M(4,3,0)$, $M(4,3,2)$	$(4, \{3, 6\}, 3)$	3
	$M(4,3,1)$	$(4, \{12, 12\}, 4)$	
	$M(4,3,3)$	$(4, \{12, 12\}, 6)$	
15	$M(5,3,0)$, $M(5,3,3)$	$(5, \{3, 15\}, 3)$	3
	$M(5,3,1)$, $M(5,3,2)$	$(5, \{15, 15\}, 4)$	
	$M(5,3,4)$	$(5, \{15, 15\}, 7)$	
18	$M(6,3,0)$, $M(6,3,4)$	$(6, \{3, 9\}, 3)$	6
	$M(6,3,1)$, $M(6,3,3)$	$(6, \{6, 18\}, 4)$	
	$M(6,3,2)$	$(6, \{9, 9\}, 5)$	
	$M(6,3,5)$	$(6, \{18, 18\}, 8)$	
	$M(3,6,0)$, $M(3,6,2)$	$(3, \{6, 18\}, 6)$	
	$M(3,6,1)$	$(3, \{18, 18\}, 7)$	

In the subsequent subsections, we proceed in a similar way. For each type DSEM M with vertex set $|V(M)|$, we construct all $M(i, j, k)$ representations by defining suitable non-homologous cycles and by finding admissible values of $i, j, k \in \mathbb{N} \cup \{0\}$. After that, we determine isomorphism maps between their representations, if exist. This gives the exact number of representations, which are maps, up to the isomorphism on that $|V(M)|$.

3.2 DSEMs of type $[3^6 : 3^3.4^2]_2$

Consider a DSEM M of type $[3^6 : 3^3.4^2]_2$ with the vertex set $V(M)$. Then the map M exists if $|V_{(3^6)}| = |V_{(3^3,4^2)}|$ (see [11]), where $|V_{(3^6)}|$ and $|V_{(3^3,4^2)}|$ are same as in Section 3.1. Now define the following three types paths in M as follows.

Definition 3.2.1 Let $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ be a path in M. We say that P_1 is of type B_1 if all the triangles incident with an inner vertex y_i lie on one side and all the quadrangles incident with y_i lie on the other side of the subpath $P'(y_{i-1}, y_i, y_{i+1})$ or every vertex of the path have face-sequence (3^6), see Figure 3.2.1. If a boundary vertex of P_1 is y_j then there is an extended path say P_{1_e} of P_1 such that y_j is an inner vertex of P_{1_e}.

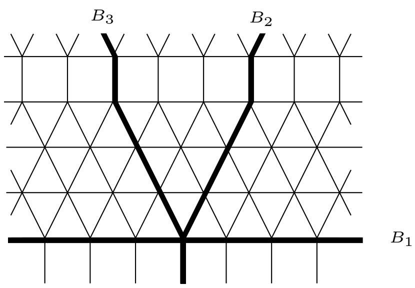

Figure 3.2.1: Paths of types B_1, B_2, B_3

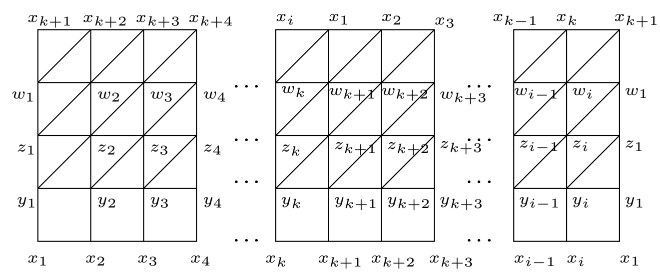

Figure 3.2.2: $M(i, 4, k)$ representation

Definition 3.2.2 Let $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ be a path in M such that z_i, z_{i+1} are inner vertices of P_2 or an extended path of P_2. We say that P_2 is of type B_2, see in Figure 3.2.1, if either of the three conditions occur for each vertex of the path.

1. if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, z_{i+1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_6(z_i, o, r, z_{i+2}, s, p)$, $\mathrm{lk}(z_{i+2}) = C_6(z_{i+1}, r, t, z_{i+3}, u, s)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{v}, z_{i+4}, \boldsymbol{w}, u, z_{i+2}, t, x)$, and if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, z_{i+1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_6(z_i, o, r, z_{i+2}, s, p)$, $\mathrm{lk}(z_{i+2}) = C_6(z_{r+1}, r, t, z_{i+3}, u, s)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{v}, z_{i+4}, \boldsymbol{w}, u, z_{i+2}, t, x)$.

2. if $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ then $\mathrm{lk}(z_{i+1}) = C_6(z_i, n, q, z_{i+2}, r, o)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{s}, z_{i+3}, \boldsymbol{t}, r, z_{i+1}, q, u)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{r}, z_{i+2}, \boldsymbol{u}, s, z_{i+4}, v, t)$, and if $\mathrm{lk}(z_i) = C_6(z_{i+1}, m, n, z_{i-1}, o, p)$, then $\mathrm{lk}(z_{i+1}) = C_6(z_{i+2}, t, m, z_i, p, q)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{r}, z_{i+3}, \boldsymbol{s}, t, z_{i+1}, q, u)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{t}, z_{i+2}, \boldsymbol{u}, r, z_{i+4}, v, s)$.

3. if $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{q}, z_{i+2}, \boldsymbol{r}, o, z_i, n, s)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{o}, z_{i+1}, \boldsymbol{s}, q, z_{i+3}, t, r)$, $\mathrm{lk}(z_{i+3}) = C_6(z_{i+2}, q, u, z_{i+4}, v, t)$, and if $\mathrm{lk}(z_i) = C_6(z_{i+1}, m, n, z_{i-1}, o, p)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{q}, z_{i+2}, \boldsymbol{r}, m, u_i, p, s)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{s}, q, z_{i+3}, t, r)$, $\mathrm{lk}(z_{i+3}) = C_6(z_{i+2}, q, u, z_{i+4}, v, t)$.

Definition 3.2.3 Let $P_3 = P(\ldots, w_{i-1}, w_i, w_{i+1}, \ldots)$ be a path in M such that w_i, w_{i+1} are inner vertices of P_3 or an extended path of P_3. We say that P_3 is of type B_3, see in Figure 3.2.1, if either of the three conditions occur for each vertex of P_3.

1. if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i-1}, \boldsymbol{n}, o, p, w_{i+1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_6(w_i, p, r, w_{i+2}, s, q)$, $\mathrm{lk}(w_{i+2}) = C_6(w_{i+1}, r, t, w_{i+3}, u, s)$, $\mathrm{lk}(w_{i+3}) = C_7(\boldsymbol{v}, w_{i+4}, \boldsymbol{w}, x, u, w_{i+2}, t)$, and if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, p, w_{i-1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{o}, w_i, \boldsymbol{q}, m, r, w_{i+2}, n)$, $\mathrm{lk}(w_{i+2}) = C_6(w_{i+1}, r, s, w_{i+3}, t, n)$, $\mathrm{lk}(w_{i+3}) = C_6(w_{i+2}, s, u, w_{i+4}, v, t)$.

2. if $\mathrm{lk}(w_i) = C_6(m, w_{i-1}, n, o, w_{i+1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_6(w_i, o, q, w_{i+2}, r, p)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+3}, \boldsymbol{t}, u, r, w_{i+1}, q)$, $\mathrm{lk}(w_{i+3}) = C_7(\boldsymbol{u}, w_{i+2}, \boldsymbol{q}, s, v, w_{i+4}, t)$, and if $\mathrm{lk}(w_i) = C_6(m, w_{i+1}, n, o, w_{i-1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_6(w_i, m, q, w_{i+2}, r, n)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+3}, \boldsymbol{t}, u, r, w_{i+1}, q)$, $\mathrm{lk}(w_{i+3}) = C_7(\boldsymbol{u}, w_{i+2}, \boldsymbol{q}, s, w_{i+4}, t)$.

3. if $\mathrm{lk}(w_i) = C_6(m, w_{i+1}, n, o, w_{i-1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{q}, w_{i+2}, \boldsymbol{r}, s, p, w_i, o)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+1}, \boldsymbol{o}, q, t, w_{i+3}, r)$, $\mathrm{lk}(w_{i+3}) = C_6(w_{i+2}, t, u, w_{i+4}, v, r)$, and if $\mathrm{lk}(w_i) = C_6(m, w_{i+1}, n, o, w_{i-1}, p)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{q}, w_{i+2}, \boldsymbol{r}, s, n, w_i, m)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{s}, w_{i+1}, \boldsymbol{m}, q, t, w_{i+3}, r)$, $\mathrm{lk}(w_{i+3}) = C_6(w_{i+2}, t, u, w_{i+4}, v, r)$.

Consider a maximal path $P(v_1, v_2, \ldots, v_i)$ of the type B_α, for $\alpha \in \{1, 2, 3\}$. By Lemmas 3.1.1, 3.1.3, there is an edge v_i-v_1 in M such that $P(v_1, v_2, \ldots, v_i) \cup \{v_i\text{-}v_1\}$ is a non-contractible cycle $Q = C_i(v_1, v_2, \ldots, v_i)$. Observe that the cycles of type B_2 and B_3 define same type cycles since they are mirror image of each other. By equation (1) of Section 3.1, there is a cycle of type B_4 in M. Let $u \in V(M)$ and Q_α be cycles of type B_α through u. As in Section 3.1, we define an $M(i, j, k)$ representation of M for some i, j, k. For this, we first cut M along the cycle Q_1 and then cut it along the cycle Q_3. Without loss of generality, suppose qudarangular faces are incident with the horizontal base cycle Q_1, see for example $M(i, 4, k)$ in Figure 3.2.2.

Lemma 3.2.1 *The DSEM M of type $[3^6 : 3^3.4^2]_2$ admits an $M(i, j, k)$ representation iff the following holds: (i) $i \geq 3$ and $j = 4m$, where $m \in \mathbb{N}$, (ii) $ij \geq 12$, (iii) $0 \leq k \leq i - 1$.*

Proof. Let M be the above type DSEM with n vertices. An $M(i, j, k)$ of M has j number of B_1 type disjoint horizontal cycles of length i. Since all the vertices of M lie in these cycles, the number of vertices in M is $n = ij$. Clearly if $i \leq 2$, M is not a map. So $i \geq 3$. If $j = 1$ then M is not a map and if $j = 2$ then M has no vertices with face-sequence (3^6). Also if $j = 2m + 1$ or $4m + 2$, where $m \in \mathbb{N}$, then we see that $|V(3^3, 4^2)| \neq |V(3^6)|$. So $j = 4m$ for $m \in \mathbb{N}$. Thus $n = ij \geq 12$. Since the length of the horizontal cycle is i, we get $0 \leq k \leq i - 1$. This completes the proof. $\square$

Let M_t, $t = 1, 2$, be DSEMs of type $[3^6 : 3^3.4^2]_2$ on n_t number of vertices and $n_1 = n_2$. Let $M_t(i_t, j_t, k_t)$ be a representation of M_t. Let $Q_{t,\alpha}$ be cycles of type B_α and $l_{t,\alpha}$ = length of the cycle of type B_α, $\alpha = 1, 2, 3, 4$, in $M_t(i_t, j_t, k_t)$. We say $M_t(i_t, j_t, k_t)$ has cycle-type $(l_{t,1}, l_{t,2}, l_{t,3}, l_{t,4})$ if $l_{t,2} \leq l_{t,3}$ or $(l_{t,1}, l_{t,3}, l_{t,2}, l_{t,4})$ if $l_{t,3} < l_{t,2}$. Now, we show the following.

Lemma 3.2.2 *The DSEMs $M_1 \cong M_2$ iff they have same cycle-type.*

Proof. Suppose M_1 and M_2 be two DSEMs of the type $[3^6 : 3^3.4^2]_2$ with same number of vertices such that they have same cycle-type. Then $l_{1,1} = l_{2,1}$, $\{l_{1,2}, l_{1,3}\} = \{l_{2,2}, l_{2,3}\}$ and $l_{1,4} = l_{2,4}$. To show $M_1 \cong M_2$, it is equivalent to show that $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Claim. $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

$M_t(i_t, j_t, k_t)$ has j_t number of horizontal cycles of type B_1, say, $Q_0 = C_{i_1}(w_{0,0}, w_{0,1}, \ldots, w_{0,i_1-1})$, $Q_1 = C_{i_1}(w_{1,0}, w_{1,1}, \ldots, w_{1,i_1-1}), \ldots, Q_{j_1-1} = C_{i_1}(w_{j_1-1,0}, w_{j_1-1,1}, \ldots, w_{j_1-1,i_1-1})$ in $M_1(i_1, j_1, k_1)$ and $Q'_0 = C_{i_2}(z_{0,0}, z_{0,1}, \ldots, z_{0,i_2-1}), Q'_1 = C_{i_2}(z_{1,0}, z_{1,1}, \ldots, z_{1,i_2-1}), \ldots, Q'_{j_2-1} = C_{i_2}(z_{j_2-1,0}, z_{j_2-1,1}, \ldots, z_{j_2-1,i_2-1})$ in $M_2(i_2, j_2, k_2)$. Then, we have the following cases.

Case 1: If $(i_1, j_1, k_1) = (i_2, j_2, k_2)$ then $i_1 = i_2$, $j_1 = j_2$, $k_1 = k_2$. Define an isomorphism $f : V(M_1(i_1, j_1, k_1)) \to V(M_2(i_2, j_2, k_2))$ such that $f(w_{u,v}) = z_{u,v}$ for $0 \leq u \leq j_1 - 1$ and $0 \leq v \leq i_1 - 1$. So by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 2: If $i_1 \neq i_2$ then $l_{1,1} \neq l_{2,1}$, a contradiction. Thus $i_1 = i_2$.

Case 3: If $j_1 \neq j_2$, then $n_1 = i_1 j_1 \neq i_2 j_2 = n_2$, as $i_1 = i_2$. A contradiction. So, $j_1 = j_2$.

Case 4: If $k_1 \neq k_2$, by assumption, $l_{1,4} = l_{2,4}$, $\mathrm{length}(Q_{1,4}) = \mathrm{length}(Q_{2,4})$. This means $\min\{k_1 + j_1, i_1 - k_1 + j_1/4\} = \min\{k_2 + j_2, i_2 - k_2 + j_2/4\}$. Since $i_1 = i_2, j_1 = j_2$ and $k_1 \neq k_2$, we see $k_1 + j_1 \neq k_2 + j_2$ and $i_1 - k_1 + j_1/4 \neq i_2 - k_2 + j_2/4$. This gives that $k_1 + j_1 = i_2 - k_2 + j_2/4 = i_1 - k_2 + j_1/4$ as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1 - k_1 - 3j_1/4$. Now identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, z_{2,0}, \ldots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,0}, z_{1,0}, z_{2,1}, z_{3,0}, \ldots, z_{j_2-1,3j_2/4-1}, z_{0,k_2+3j_2/4})$ of type B_2 through vertex $z_{0,0}$. This gives another representation of M_2, say R, with a map $f' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that $f'(z_{u,v}) = z_{u,(i_2-v+\lfloor 3u/4\rfloor)(mod\, i_2)}$ for $0 \le u \le j_2-1$ and $0 \le v \le i_2-1$. The lower and upper horizontal cycles of R are $Q' = C_{i_2}(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \ldots, z_{0,1})$ and $Q'' = C_{i_2}(z_{0,k_2+3j_2/4}, z_{0,k_2+3j_2/4-1}, \ldots, z_{0,k_2+3j_2/4+1}$) respectively. The path $P(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \ldots, z_{0,k_2+3j_2/4})$ in Q' has length $i_2 - k_2 - 3j_2/4$. Note that R has j_2 number of horizontal cycle of length i_2. So, $R = M_2(i_2, j_2, i_2 - k_2 - 3j_2/4)$. Now $i_2 - k_2 - 3j_2/4 = i_2 - (i_1 - k_1 - 3j_1/4) - 3j_2/4 = k_1$ as $i_2 = i_1$ and $j_2 = j_1$. Thus, $(i_2, j_2, i_2 - k_2 - 3j_2/4)) = (i_1, j_1, k_1)$ and hence by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$. By cases 1-4, claim follows and therefore $M_1 \cong M_2$.

Suppose $M_1 \cong M_2$. Proceeding similarly as in Lemma 3.1.7, we get $l_{1,1} = l_{2,1}$, $\{l_{1,2}, l_{1,3}\} = \{l_{2,2}, l_{2,3}\}$ and $l_{1,4} = l_{2,4}$. Hence, M_1 and M_2 have same cycle-type. $\square$

Now doing the computation for the first four admissible values of $|V(M)|$, we get Table 3.2. For $|V(M)| = 12$, we illustrate the computation as follows.

Example 3.2.1 Let M be a DSEM of type $[3^6 : 3^3.4^2]_2$ with 12 vertices on the torus. By Lemma 3.2.1, M has three $M(i, j, k)$ representation, namely, $M(3, 4, 0), M(3, 4, 1)$ and $M(3, 4, 2)$, see Figures 3.2.3, 3.2.4, 3.2.5 respectively. In $M(3, 4, 0)$, $Q_{1,1} = C_3(v_1, v_2, v_3)$ is a B_1 type cycle, $Q_{1,2} = C_4(v_1, v_4, u_2, u_6)$ and $Q_{1,3} = C_4(v_1, v_4, u_1, u_4)$ are two B_2 type cycles and $Q_{1,4} = C_4(v_1, v_4, u_1, u_4)$ is a B_4 type cycle. In $M(3, 4, 1)$, $Q_{2,1} = C_3(w_1, w_2, w_3)$ is a B_1 type cycle, $Q_{2,2} = C_{12}(w_1, w_4, u_2, u_6, w_2, w_5, u_3, u_4, w_3, w_6, u_1, u_5)$ and $Q_{2,3} = C_{12}(w_1, w_4, u_1, u_4, w_2, w_5, u_2, u_5, w_3, w_6, u_3, u_6)$ are two B_2 type cycles and $Q_{2,4} = C_5(w_2, w_5, u_2, u_5, w_3)$ is a B_4 type cycles. In $M(3, 4, 2)$, $Q_{3,1} = C_3(x_1, x_2, x_3)$ is a B_1 type cycle, $Q_{3,2} = C_{12}(x_1, x_4, u_2, u_6, x_3, x_6, u_1, u_5, x_2, x_5, u_3, u_4)$ and $Q_{3,3} = C_{12}(x_1, x_4, u_1, u_4, x_3, x_6, u_3, u_6, x_2, x_5, u_2, u_5)$ are B_2 type cycles and $Q_{3,4} = C_5(x_3, x_6, u_1, u_5, x_2)$ is a B_4 type cycle.

In $M(i, j, k)$, observe that B_1 type cycles have the same length and B_2 type cycles have at most two different lengths. Since length$(Q_{1,4}) \ne$ length$(Q_{r,4})$ for $r = 2, 3$, $M(3, 4, 0) \ncong M(3, 4, 1)$, $M(3, 4, 2)$. Observe that, length $(Q_{2,1}) =$ length$(Q_{3,1})$, {length$(Q_{2,2})$, length$(Q_{2,3})$} = {length$(Q_{3,2})$, length$(Q_{3,3})$} and length$(Q_{2,4}) =$ length$(Q_{3,4})$. Now cutting $M(3, 4, 1)$ along the path $P(w_2, w_5, u_3, u_4, w_3)$ and identifying along the path $P(w_1, w_4, u_1, u_4, w_2)$, leads to Figure 3.2.6, i.e., $M(3, 4, 2)$. By Lemma 3.2.2, $M(3, 4, 1) \cong M(3, 4, 2)$. Therefore, there are two DSEMs of type $[3^6 : 3^3.4^2]_2$ with 12 vertices on the torus upto isomorphism.

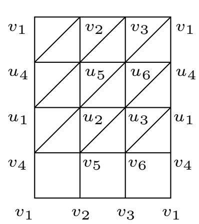

Figure 3.2.3 : $M(3, 4, 0)$

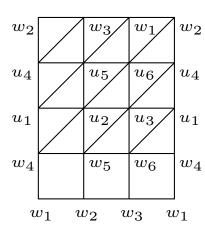

Figure 3.2.4 : $M(3, 4, 1)$

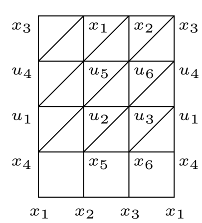

Figure 3.2.5 : $M(3, 4, 2)$

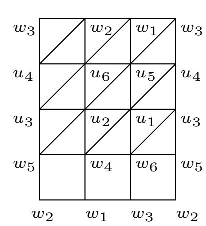

Figure 3.2.6 : $M(3, 4, 2)$

Table 3.2 : DSEMs of type $[3^6 : 3^3.4^2]_2$ on the torus for $|V(M)| \le 24$

$\lvert V(M)\rvert$	Isomorphic classes	Length of cycles	No. of maps
12	$M(3, 4, 0)$	$(3, \{4, 4\}, 4)$	2
	$M(3, 4, 1)$, $M(3, 4, 2)$	$(3, \{12, 12\}, 5)$	
16	$M(4, 4, 0)$, $M(4, 4, 1)$	$(4, \{4, 16\}, 4)$	2
	$M(4, 4, 2)$, $M(4, 4, 3)$	$(4, \{8, 16\}, 6)$	
20	$M(5, 4, 0)$, $M(5, 4, 2)$	$(5, \{4, 20\}, 4)$	3
	$M(5, 4, 3)$, $M(5, 4, 4)$	$(5, \{20, 20\}, 7)$	
	$M(5, 4, 1)$	$(5, \{20, 20\}, 5)$	

24	$M(6,4,0)$, $M(6,4,3)$	$(6,\{4,8\},4)$	5
	$M(6,4,1)$, $M(6,4,2)$	$(6,\{12,24\},5)$	
	$M(6,4,4)$, $M(6,4,5)$	$(6,\{12,24\},8)$	
	$M(3,8,0)$	$(3,\{8,8\},8)$	
	$M(3,8,1)$, $M(3,8,2)$	$(3,\{24,24\},9)$	

3.3 DSEMs of type $[3^3.4^2 : 4^4]_1$

Consider a DSEM M of type $[3^3 : 4^2.4^4]_1$ with the vertex set $V(M)$. By [11], we get $|V_{(3^3,4^2)}| = 2|V_{(4^4)}|$, where $|V_{(3^3,4^2)}|$ and $|V_{(4^4)}|$ denote the cardinality of vertex sets $V_{(3^3,4^2)}$ and $V_{(4^4)}$ respectively. Define the following three types of paths in M as follows.

Definition 3.3.1 Consider a path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M. We say P_1 is of type X_1 if all the triangles incident with an inner vertex y_i lie on one side and all the quadrangles incident with y_i lie on the other side of the subpath $P'(y_{i-1}, y_i, y_{i+1})$ or every vertices of the path have face-sequence (4^4), see in Figure 3.3.1. If a boundary vertex of P_1 is y_j then there is an extended path say P_{1_e} of P_1 such that y_j is an inner vertex of P_{1_e}.

Definition 3.3.2 Consider a path $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M such that z_i, z_{i+1} are inner vertices of P_2 or an extended path of P_2. We say that P_2 is of type X_2, see in Figure 3.3.1, if either of the two conditions occur for each vertex of the path.

1. if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, z_{i+1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{r}, z_{i+2}, \boldsymbol{s}, p, z_i, o, t)$, $\mathrm{lk}(z_{i+2}) = C_8(\boldsymbol{u}, z_{i+3}, \boldsymbol{v}, s, \boldsymbol{p}, z_{i+1}, \boldsymbol{t}, r)$, and if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, z_{i-1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_8(\boldsymbol{o}, z_i, \boldsymbol{q}, m, \boldsymbol{r}, z_{i+2}, \boldsymbol{s}, n)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{n}, z_{i+1}, \boldsymbol{m}, r, z_{i+3}, t, s)$.
2. if $\mathrm{lk}(z_i) = C_8(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, \boldsymbol{p}, z_{i+1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{r}, z_i, \boldsymbol{o}, p, z_{i+2}, s, q)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{t}, z_{i+3}, \boldsymbol{u}, s, z_{i+1}, p, v)$, and if $\mathrm{lk}(z_i) = C_8(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, \boldsymbol{p}, z_{i-1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{o}, z_i, \boldsymbol{r}, m, z_{i+2}, s, n)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{t}, z_{i+3}, \boldsymbol{u}, s, z_{i+1}, m, v)$.

Definition 3.3.3 Consider a path $P_3 = P(\ldots, w_{i-1}, w_i, w_{i+1}, \ldots)$ in M such that w_i, w_{i+1} are inner vertices of P_3 or an extended path of P_3. We say that P_3 is of type X_3, see in Figure 3.3.1, if either of the two conditions occur for each vertex of the path.

1. if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i-1}, \boldsymbol{n}, o, p, w_{i+1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{r}, w_{i+2}, \boldsymbol{s}, t, q, w_i, p)$, $\mathrm{lk}(w_{i+2}) = C_8(\boldsymbol{t}, w_{i+1}, \boldsymbol{p}, r, \boldsymbol{u}, w_{i+3}, \boldsymbol{v}, s)$, and if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, p, w_{i-1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_8(\boldsymbol{o}, w_i, \boldsymbol{q}, m, \boldsymbol{r}, w_{i+2}, \boldsymbol{s}, n)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{n}, w_{i+1}, \boldsymbol{m}, r, t, w_{i+3}, s)$.
2. if $\mathrm{lk}(w_i) = C_8(\boldsymbol{m}, w_{i-1}, \boldsymbol{n}, o, \boldsymbol{p}, w_{i+1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{r}, w_i, \boldsymbol{o}, p, s, w_{i+2}, q)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{t}, w_{i+3}, \boldsymbol{w}, v, q, w_{i+1}, s)$, and if $\mathrm{lk}(w_i) = C_8(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, \boldsymbol{p}, w_{i-1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{o}, w_i, \boldsymbol{r}, m, s, w_{i+2}, n)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{t}, w_{i+3}, \boldsymbol{w}, v, n, w_{i+1}, s)$.

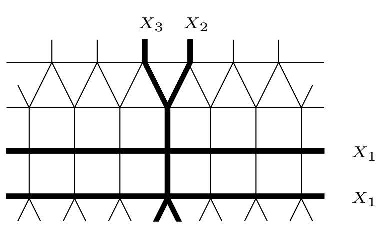

Figure 3.3.1: Paths of types X_1, X_2, X_3

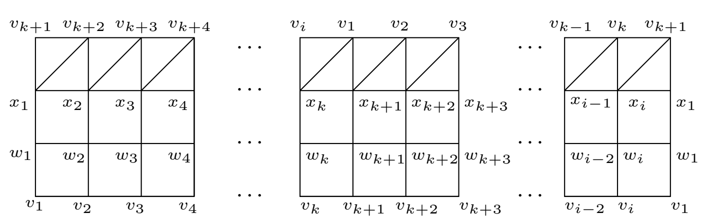

Figure 3.3.2: $M(i, 3, k)$

As in Section 3.1, for a maximal path P of the type X_α, $\alpha \in \{1, 2, 3\}$ there is an edge e in M such that $P \cup e$ is a non-contractible cycle of respective type. The cycles of types X_2 and X_3 define same type of cycles as they are mirror image of each other. Following equation (1) of section 3.1, there is a cycle of type X_4 in M. As in Section 3.1, we define an $M(i,j,k)$ representation for M for some i,j,k. See for example $M(i,3,k)$ in Figure 3.3.2.

Lemma 3.3.1 *The DSEM M of type $[3^3.4^2 : 4^4]_1$ admits an $M(i,j,k)$-representation iff the following holds: (i) $i \geq 3$ and $j = 3m$, where $m \in \mathbb{N}$, (ii) $ij \geq 9$, (iii) $0 \leq k \leq i-1$.*

Proof. Let M be a DSEM of type $[3^3.4^2 : 4^4]_1$ with n vertices. Observe that the map exists if $|V(3^3,4^2)| = 2|V(4^4)|$. Now proceeding, as in Lemma 3.1.6, we get all possible values of i, j and k of $M(i,j,k)$. Thus the proof. □

For $t = 1,2$, let M_t be DSEMs of type $[3^3.4^2 : 4^4]_1$ on n_t number of vertices with $n_1 = n_2$. Let $M_t(i_t,j_t,k_t)$ be a representation of M_t and $Q_{t,\alpha}$ be cycles of type X_α, $\alpha = 1,2,3,4$. If $l_{t,\alpha}$ = length of the cycle of type X_α in $M_t(i_t,j_t,k_t)$ then we say that $M_t(i_t,j_t,k_t)$ has cycle-type $(l_{t,1}, l_{t,2}, l_{t,3}, l_{t,4})$ if $l_{t,2} \leq l_{t,3}$ or $(l_{t,1}, l_{t,3}, l_{t,2}, l_{t,4})$ if $l_{t,3} < l_{t,2}$. Now, we show the following lemma.

Lemma 3.3.2 *The DSEMs $M_1 \cong M_2$ iff they have same cycle-type.*

Proof. Suppose M_1 and M_2 be two DSEMs of the type $[3^3.4^2 : 4^4]_1$ with same cardinality such that they have same cycle-type. This means $a_{1,1} = a_{2,1}$, $\{l_{1,2}, l_{1,3}\} = \{l_{2,2}, l_{2,3}\}$ and $l_{1,4} = l_{2,4}$.
Claim. $M_1(i_1,j_1,k_1) \cong M_2(i_2,j_2,k_2)$.

By the definition, $M_t(i_t,j_t,k_t)$ has j_t number of X_1 type disjoint horizontal cycles of length i_t, say, $Q_0 = C_{i_1}(w_{0,0}, w_{0,1}, \ldots, w_{0,i_1-1}), Q_1 = C_{i_1}(w_{1,0}, w_{1,1}, \ldots, w_{1,i_1-1}), \ldots, Q_{j_1-1} = C_{i_1}(w_{j_1-1,0}, w_{j_1-1,1}, \ldots, w_{j_1-1,i_1-1})$ in $M_1(i_1,j_1,k_1)$ and $Q'_0 = C_{i_2}(z_{0,0}, z_{0,1}, \ldots, z_{0,i_2-1})$, $Q'_1 = C_{i_2}(z_{1,0}, z_{1,1}, \ldots, z_{1,i_2-1}), \ldots, Q'_{j_2-1} = C_{i_2}(z_{j_2-1,0}, z_{j_2-1,1}, \ldots, z_{j_2-1,i_2-1})$ in $M_2(i_2,j_2,k_2)$. Then,
Case 1: If $(i_1,j_1,k_1) = (i_2,j_2,k_2)$ then $i_1 = i_2, j_1 = j_2, k_1 = k_2$. Define an isomorphism $f : V(M_1(i_1,j_1,k_1)) \to V(M_2(i_2,j_2,k_2))$ such that $f(w_{u,v}) = z_{u,v}$ for $0 \leq u \leq j_1-1$ and $0 \leq v \leq i_1-1$. So by f, $M_1(i_1,j_1,k_1) \cong M_2(i_2,j_2,k_2)$.
Case 2: If $i_1 \neq i_2$ then contradicting the fact that $l_{1,1} = l_{2,1}$. Thus $i_1 = i_2$.
Case 3: If $j_1 \neq j_2$, it implies that $n_1 = i_1j_1 \neq n_2 = i_2j_2$ as $i_1 = i_2$. A contradiction. So, $j_1 = j_2$.
Case 4: If $k_1 \neq k_2$ then by $l_{1,4} = l_{2,4}$ we see length$(Q_{1,4})$ = length$(Q_{2,4})$. This means $\min\{k_1 + j_1, j_1 + (i_1 - k_1 - j_1/3)\} = \min\{k_2 + j_2, j_2 + (i_2 - k_2 - j_2/3)\}$. Since $i_1 = i_2, j_1 = j_2$ and $k_1 \neq k_2$, $k_1 + j_1 \neq k_2 + j_2$ and $j_1 + (i_1 - k_1 - j_1/3) \neq j_2 + (i_2 - k_2 - j_2/3)$. This implies $k_1 + j_1 = i_2 - k_2 - j_2/3 + j_2 = i_1 - k_2 + 2j_1/3$ as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1 - k_1 - j_1/3$. Now identify $M_2(i_2,j_2,k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, \ldots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,0}, z_{1,0}, z_{2,0}, z_{3,1}, \ldots, z_{j_2-1,j_2/3-1}, z_{0,k_2+j_2/3})$ of type X_2 through vertex $z_{0,0}$. This gives another representation of M_2, say R, with a map $f' : V(M_2(i_2,j_2,k_2)) \to V(R)$ such that $f'(z_{u,v}) = z_{u,(i_2-v+\lfloor t/3\rfloor)(mod\, i_2)}$ for $0 \leq u \leq j_2-1$ and $0 \leq v \leq i_2-1$. In R the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \ldots, z_{0,1})$ and $Q'' = C_{i_2}(z_{0,k_2+j_2/3}, z_{0,k_2+j_2/3-1}, \ldots, z_{0,k_2+j_2/3+1})$ respectively. The path $P(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \ldots, z_{0,k_2+j_2/3})$ in Q' has length $i_2 - k_2 - j_2/3$. Note that R has j_2 number of horizontal cycles of length i_2. So, $R = M_2(i_2, j_2, i_2 - k_2 - j_2/3)$. Note that $i_2 - k_2 - j_2/3 = i_2 - (i_1 - k_1 - j_1/3) - j_2/3 = k_1$ as $i_2 = i_1, j_2 = j_1$. Thus, $(i_2, j_2, i_2 - k_2 - j_2/3) = (i_1, j_1, k_1)$. Therefore, $M_1(i_1,j_1,k_1) \cong M_2(i_2,j_2,k_2)$. So the claim follows. Hence, $M_1 \cong M_2$.

The converse part follows from the converse part of Lemma 3.1.7. This completes the proof. □

Now computing the DSEMs for the first four admissible values of $|V(M)|$, we get Table 3.3. We illustrate this computation below for $|V(M)| = 9$.

Example 3.3.1 Let M be a DSEM of type $[3^3.4^2 : 4^4]_1$ with 9 vertices on the torus. By Lemma 3.3.1, M has three $M(i,j,k)$ representation, namely, $M(3,3,0), M(3,3,1)$ and $M(3,3,2)$, see Figures 3.3.3, 3.3.4, 3.3.5 respectively. In $M(3,3,0)$, $Q_{1,1} = C_3(x_1,x_2,x_3)$ is a X_1 type cycle, $Q_{1,2} =$

$C_9(x_1, x_4, x_7, x_2, x_5, x_8, x_3, x_6, x_9)$ and $Q_{1,3} = C_3(x_1, x_4, x_7)$ are X_2 type cycles and $Q_{1,4} = C_3(x_1, x_4, x_7)$ is a X_4 type cycle. In $M(3, 3, 1)$, $Q_{2,1} = C_3(y_1, y_2, y_3)$ is a X_1 type cycle, $Q_{2,2} = C_9(y_1, y_4, y_7, y_3, y_6, y_9, y_2, y_5, y_8)$ and $Q_{2,3} = C_9(y_1, y_4, y_7, y_2, y_5, y_8, y_3, y_6, y_9)$ are X_2 type cycles and $Q_{2,4} = C_4(y_2, y_5, y_8, y_1)$ is a X_4 type cycle. In $M(3, 3, 2)$, $Q_{3,1} = C_3(z_1, z_2, z_3)$ is a X_1 type cycle, $Q_{3,2} = C_3(z_1, z_4, z_7)$ and $Q_{3,3} = C_9(z_1, z_4, z_7, z_3, z_6, z_9, z_2, z_5, z_8)$ are X_2 type cycle and $Q_{3,4} = C_3(z_3, z_6, z_9)$ is a X_4 type cycle.

Observe that type X_1 cycles have the same length and type X_2 cycles have at most two different lengths. Since length$(Q_{2,4}) \neq$ length$(Q_{r,4})$ for $r = 1, 3$, $M(3, 3, 1) \ncong M(3, 3, 0), M(3, 3, 2)$. Observe that, length $(Q_{1,1})$ = length$(Q_{3,1})$, {length$(Q_{1,2})$, length$(Q_{1,3})$} = { length$(Q_{3,2})$, length$(Q_{3,3})$} and length$(Q_{1,4})$ =length$(Q_{3,4})$. Now identifying the vertical boundary of $M(3, 3, 0)$ and cutting along the path $P(x_1, x_4, x_7, x_2)$ leads to Figure 3.3.6, i.e., $M(3, 3, 2)$. By the isomorphism map define in Lemma 3.3.2, $M(3, 3, 0) \cong M(3, 3, 2)$. Therefore, there are two DSEMs of type $[3^3.4^2 : 4^4]_1$ with 9 vertices on the torus upto isomorphism.

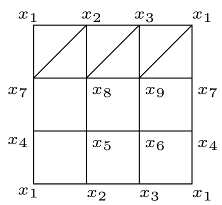

Figure 3.3.3: $M(3, 3, 0)$

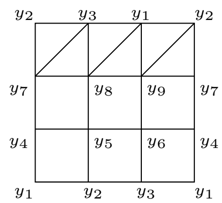

Figure 3.3.4: $M(3, 3, 1)$

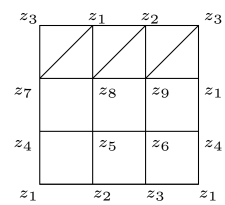

Figure 3.3.5: $M(3, 3, 2)$

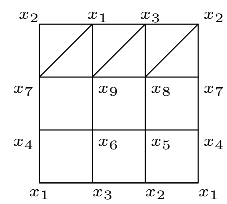

Figure 3.3.6: $M(3, 3, 2)$

Table 3.3: DSEMs of type $[3^3.4^2 : 4^4]_1$ on the torus for $|V(M)| \leq 18$

$\lvert V(M)\rvert$	Isomorphic classes	Length of cycles	No of maps
9	$M(3, 3, 0)$, $M(3, 3, 2)$	$(3, \{3, 9\}, 3)$	2
	$M(3, 3, 1)$	$(3, \{9, 9\}, 4)$	
12	$M(4, 3, 0)$, $M(4, 3, 3)$	$(4, \{3, 12\}, 3)$	2
	$M(4, 3, 1)$, $M(4, 3, 2)$	$(4, \{6, 12\}, 4)$	
15	$M(5, 3, 0)$, $M(5, 3, 4)$	$(5, \{3, 15\}, 3)$	3
	$M(5, 3, 1)$, $M(5, 3, 3)$	$(5, \{15, 15\}, 4)$	
	$M(5, 3, 2)$	$(5, \{15, 15\}, 5)$	
18	$M(6, 3, 0)$, $M(6, 3, 5)$	$(6, \{3, 18\}, 3)$	5
	$M(6, 3, 1)$, $M(6, 3, 4)$	$(6, \{9, 18\}, 4)$	
	$M(6, 3, 2)$, $M(6, 3, 3)$	$(6, \{6, 9\}, 5)$	
	$M(3, 6, 0)$, $M(3, 6, 1)$	$(3, \{6, 18\}, 6)$	
	$M(3, 6, 2)$	$(3, \{18, 18\}, 8)$	

3.4 DSEMs of type $[3^3.4^2 : 4^4]_2$

Consider a DSEM M of type $[3^3.4^2 : 4^4]_2$ with vertex set $V(M)$. By [11], we have $|V_{(4^4)}| = |V_{(3^3,4^2)}| = 2k$ for some $k \in \mathbb{N}$, where $|V_{(4^4)}|$ and $|V_{(3^3,4^2)}|$ are same as in Section 3.3. Now define following three types of paths in M.

Definition 3.4.1 A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M is of type Y_1 if all the triangles incident with an inner vertex y_i lie on one side and all the quadrangles incident with y_i lie on the other side of the subpath $P'(y_{i-1}, y_i, y_{i+1})$ or every vertex of the path have face-sequence (4^4), see in Figure 3.4.1. If a boundary vertex of P_1 is y_j then there is an extended path say P_{1_e} of P_1 such that y_j is an inner vertex of P_{1_e}.

Definition 3.4.2 Consider a path $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M such that z_i, z_{i+1} are inner vertices of P_2 or an extended path of P_2. We say that P_2 is of type Y_2, see in Figure 3.4.1, if either of the two conditions occur for each vertex of the path.

1. if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, z_{i+1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{r}, z_{i+2}, \boldsymbol{s}, p, z_i, o, t)$, $\mathrm{lk}(z_{i+2}) = C_8(\boldsymbol{u}, z_{i+3}, \boldsymbol{v}, s, \boldsymbol{p}, z_{i+1}, \boldsymbol{t}, r)$, $\mathrm{lk}(z_{i+3}) = C_8(\boldsymbol{w}, z_{i+4}, \boldsymbol{x}, v, \boldsymbol{s}, z_{i+2}, \boldsymbol{r}, u)$, and if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, z_{i-1}, p, q)$ then $\mathrm{lk}(z_{i+1}) = C_8(\boldsymbol{o}, z_i, \boldsymbol{q}, m, \boldsymbol{r}, z_{i+2}, \boldsymbol{s}, n)$, $\mathrm{lk}(z_{i+2}) = C_8(\boldsymbol{n}, z_{i+1}, \boldsymbol{m}, r, \boldsymbol{t}, z_{i+3}, \boldsymbol{u}, s)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{s}, z_{i+2}, \boldsymbol{r}, t, z_{i+4}, v, u)$.

2. if $\mathrm{lk}(z_i) = C_8(\boldsymbol{m}, u_{i-1}, \boldsymbol{n}, o, \boldsymbol{p}, z_{i+1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{r}, z_i, \boldsymbol{o}, p, z_{i+2}, s, q)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{t}, z_{i+3}, \boldsymbol{u}, s, z_{i+1}, p, v)$, $\mathrm{lk}(z_{i+3}) = C_8(\boldsymbol{s}, z_{i+2}, \boldsymbol{v}, t, \boldsymbol{w}, z_{i+4}, \boldsymbol{x}, u)$, and if $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, \boldsymbol{p}, u_{i-1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(z_{i+1}) = C_8(\boldsymbol{o}, z_i, \boldsymbol{r}, m, \boldsymbol{s}, z_{i+2}, \boldsymbol{t}, n)$, $\mathrm{lk}(z_{i+2}) = C_8(\boldsymbol{n}, z_{i+1}, \boldsymbol{m}, s, z_{i+3}, u, t)$, $\mathrm{lk}(z_{i+3}) = C_7(\boldsymbol{w}, z_{i+4}, \boldsymbol{x}, u, z_{i+2}, \boldsymbol{s}, v)$.

Definition 3.4.3 Consider a path $P_3 = P(\ldots, w_{i-1}, w_i, w_{i+1}, \ldots)$ in M such that w_i, w_{i+1} are inner vertices of P_3 or an extended path of P_3. We say that P_3 is of type Y_3, see in Figure 3.4.1, if either of the two conditions occur for each vertex of the path.

1. if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, u_{i-1}, \boldsymbol{n}, o, p, w_{i+1}, q)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{r}, w_{i+2}, \boldsymbol{s}, t, q, w_i, p)$, $\mathrm{lk}(w_{i+2}) = C_8(\boldsymbol{t}, w_{i+1}, \boldsymbol{p}, r, \boldsymbol{u}, w_{i+3}, \boldsymbol{v}, s)$, $\mathrm{lk}(w_{i+3}) = C_8(\boldsymbol{w}, w_{i+4}, \boldsymbol{x}, v, \boldsymbol{s}, w_{i+2}, \boldsymbol{r}, u)$, and if $\mathrm{lk}(w_i) = C_7(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, p, u_{i-1}, q)$ then $lk(w_{i+1}) = C_8(\boldsymbol{o}, w_i, \boldsymbol{q}, a, \boldsymbol{r}, w_{i+2}, \boldsymbol{s}, n)$, $\mathrm{lk}(w_{i+2}) = C_8(\boldsymbol{n}, w_{i+1}, \boldsymbol{m}, r, \boldsymbol{t}, w_{i+3}, \boldsymbol{u}, s)$, $\mathrm{lk}(w_{i+3}) = C_7(\boldsymbol{s}, w_{i+2}, \boldsymbol{r}, t, v, w_{i+4}, u)$.

2. if $\mathrm{lk}(w_i) = C_8(\boldsymbol{m}, u_{i-1}, \boldsymbol{n}, o, \boldsymbol{p}, w_{i+1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(w_{i+1}) = C_7(\boldsymbol{r}, w_i, \boldsymbol{o}, p, s, w_{i+2}, q)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{t}, w_{i+3}, \boldsymbol{u}, v, q, w_{i+1}, s)$, $\mathrm{lk}(w_{i+3}) = C_8(\boldsymbol{w}, w_{i+4}, \boldsymbol{x}, u, \boldsymbol{v}, w_{i+2}, \boldsymbol{s}, t)$, and if $\mathrm{lk}(w_i) = C_8(\boldsymbol{m}, w_{i+1}, \boldsymbol{n}, o, \boldsymbol{p}, u_{i-1}, \boldsymbol{q}, r)$ then $\mathrm{lk}(w_{i+1}) = C_8(\boldsymbol{o}, w_i, \boldsymbol{r}, m, \boldsymbol{s}, w_{i+2}, \boldsymbol{t}, n)$, $\mathrm{lk}(w_{i+2}) = C_7(\boldsymbol{n}, w_{i+1}, \boldsymbol{m}, s, u, w_{i+3}, t)$, $\mathrm{lk}(w_{i+3}) = C_7(\boldsymbol{v}, w_{i+4}, \boldsymbol{w}, x, t, w_{i+2}, u)$.

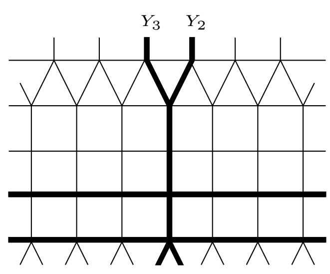

Figure 3.4.1: Paths of type Y_1, Y_2, Y_3

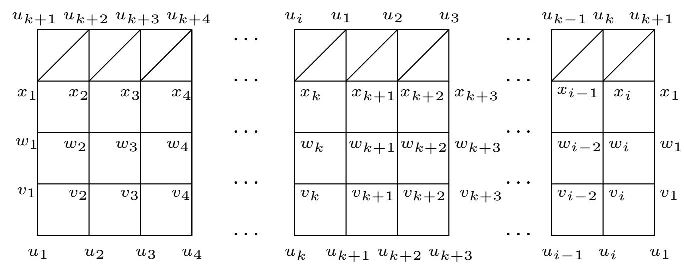

Figure 3.4.2: $M(i, 4, k)$ representation

Consider a maximal path $P(v_1, v_2, \ldots, v_i)$ of the type Y_α, $\alpha \in \{1, 2, 3\}$. Following Lemmas 3.1.1, 3.1.3, there is an edge v_i-v_1 in M such that $P(v_1, v_2, \ldots, v_i) \cup \{v_i\text{-}v_1\}$ is a non-contractible cycle $Q = C_i(v_1, v_2, \ldots, v_i)$. The cycles of type Y_2 and Y_3 define same type cycles as they are mirror image of each other. By equation (1) of Section 3.1, define a cycle of type Y_4 in M. Let $u \in V(M)$ and Q_α be cycles of type Y_α through u. As in Section 3.1, we define an $M(i, j, k)$ representation of M for some i, j, k. For this, we first cut M along the cycle Q_1 and then cut it along the cycle Q_3. Without loss of generality, assume that the faces incident on the base horizontal cycle are quadrangular, for examples see $M(i, 4, k)$ in Figure 3.4.2. Then we prove the following lemma.

Lemma 3.4.1 *The DSEM M of type $[3^3.4^2 : 4^4]_2$ admits an $M(i, j, k)$ representation iff the following holds: (i) $i \geq 3$ and $j = 4m$, where $m \in \mathbb{N}$, (ii) $ij \geq 12$, (iii) $0 \leq k \leq i - 1$.*

Proof. Let M be a DSEM of type $[3^3.4^2 : 4^4]_2$ with n vertices. Then for the existence of M we have $|V(3^3, 4^2)| = |V(4^4)|$. Now proceeding as in Lemma 3.2.1, we get the result. $\square$

For $t = 1, 2$, let M_t be DSEMs of type $[3^3.4^2 : 4^4]_2$ on n_t number of vertices with representation $M_t(i_t, j_t, k_t)$. Suppose $n_1 = n_2$. Let $Q_{t,\alpha}$ be the cycles of type Y_α and $l_{t,\alpha}$ = length of the cycle of type Y_α, for $\alpha = 1, 2, 3, 4$, in $M_t(i_t, j_t, k_t)$. We say that $M_t(i_t, j_t, k_t)$ has cycle-type $(l_{t,1}, l_{t,2}, l_{t,3}, l_{t,4})$ if $l_{t,2} \leq l_{t,3}$ or $(l_{t,1}, l_{t,3}, l_{t,2}, l_{t,4})$ if $l_{t,3} < l_{t,2}$. Now, we show the following lemma.

Lemma 3.4.2 *The DSEMs $M_1 \cong M_2$ if and only if they have same cycle-type.*

Proof. Suppose M_1 and M_2 be two DSEMs of the type $[3^3.4^2 : 4^4]_2$ with same number of vertices such that they have same cycle-type. Then $l_{1,1} = l_{2,1}$, $\{l_{1,2}, l_{1,3}\} = \{l_{2,2}, l_{2,3}\}$ and $l_{1,4} = l_{2,4}$. Claim that $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Note that $M_t(i_t, j_t, k_t)$ has j_t number of Y_1 types disjoint horizontal cycles of length i_t, say $Q_0 = C_{i_1}(y_{0,0}, y_{0,1}, \dots, y_{0,i_1-1}), Q_1 = C_{i_1}(y_{1,0}, y_{1,1}, \dots, y_{1,i_1-1}), \dots, Q_{j_1-1} = C_{i_1}(y_{j_1-1,0}, y_{j_1-1,1}, \dots, y_{j_1-1,i_1-1})$ in $M_1(i_1, j_1, k_1)$ and $Q'_0 = C_{i_2}(z_{0,0}, z_{0,1}, \dots, z_{0,i_2-1}), Q'_1 = C_{i_2}(z_{1,0}, z_{1,1}, \dots, z_{1,i_2-1}), \dots, Q'_{j_2-1} = C_{i_2}(z_{j_2-1,0}, z_{j_2-1,1}, \dots, z_{j_2-1,i_2-1})$ in $M_2(i_2, j_2, k_2)$. Then we have the following cases.

Case 1: If $(i_1, j_1, k_1) = (i_2, j_2, k_2)$ then $i_1 = i_2$, $j_1 = j_2$ and $k_1 = k_2$. Define an isomorphism $f : V(M_1(i_1, j_1, k_1)) \to V(M_2(i_2, j_2, k_2))$ such that $f(y_{u,v}) = z_{u,v}$ for $0 \le u \le j_1 - 1$ and $0 \le v \le i_1 - 1$. So, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 2: If $i_1 \ne i_2$ then it contradicting the fact $l_{1,1} = l_{2,1}$. So $i_1 = i_2$.

Case 3: If $j_1 \ne j_2$ then it contradicting the fact that $n_1 \ne n_2$. So $j_1 = j_2$.

Case 4: If $k_1 \ne k_2$ then by, $l_{1,4} = l_{2,4}$, we see length$(Q_{1,4})$ = length $(Q_{2,4})$. This implies $\min\{k_1 + j_1, j_1 + (i_1 - k_1 - j_1/4)\} = \min\{k_2 + j_2, j_2 + (i_2 - k_2 - j_2/4)\}$. Since $i_1 = i_2, j_1 = j_2$ and $k_1 \ne k_2$, it follows that $k_1 + j_1 \ne k_2 + j_2$ and $j_1 + (i_1 - k_1 - j_1/4) \ne j_2 + (i_2 - k_2 - j_2/4)$. This gives $k_1 + j_1 = i_2 - k_2 - j_2/4 + j_2 = i_1 - k_2 + 3j_1/4$ as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1 - k_1 - j_1/4$. Now identify $M_2(i_2, j_2, k_2)$ along the path $P(z_{0,0}, z_{1,0}, \dots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,0}, z_{1,0}, z_{2,0}, z_{3,0}, z_{4,1}, \dots, z_{j_2-1,j_2/4-1}, z_{0,k_2+j_2/4})$ of type Y_2 through vertex $z_{0,0}$. This gives another representation of M_2, say R, with a map $f' : V(T_2) \to V(R)$ such that $f'(z_{u,v}) = z_{u,(i_2-v+\lfloor u/4 \rfloor)(mod\, i_2)}$ for $0 \le u \le j_2-1$ and $0 \le v \le i_2-1$. In R the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \dots, z_{0,1})$ and $Q'' = C_{i_2}(z_{0,k_2+j_2/4}, z_{0,k_2+j_2/4-1}, \dots, z_{0,k_2+j_2/4+1})$ respectively. The path $P(z_{0,0}, z_{0,i_2-1}, z_{0,i_2-2}, \dots, z_{0,k_2+j_2/4})$ in Q' has length $i_2 - k_2 - j_2/4$. Note that R has j_2 number of horizontal cycle of length i_2. So, $R = M_2(i_2, j_2, i_2 - k_2 - j_2/4)$. Also $i_2 - k_2 - j_2/4 = i_2 - (i_1 - k_1 - j_1/4) - j_2/4 = k_1$ since $i_2 = i_1, j_2 = j_1$. Thus, $(i_2, j_2, i_2 - k_2 - j_2/4) = (i_1, j_1, k_1)$. Therefore by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$ and hence, $M_1 \cong M_2$.

In the same way as in the converse part of the Lemma 3.1.7, we get the converse part of the above lemma. This completes the proof. $\square$

Now computing the DSEMs for the first four admissible values of $|V(M)|$, we get Table 3.4. We illustrate this computation for $|V(M)| = 12$ as follows.

Example 3.4.1 Let M be a DSEM of type $[3^3.4^2 : 4^4]_2$ with 12 vertices on the torus. By Lemma 3.4.1, M has three $M(i,j,k)$ representation, namely, $M(3,4,0), M(3,4,1)$ and $M(3,4,2)$, see Figures 3.4.3, 3.4.4, 3.4.5 respectively. In $M(3,4,0)$, $Q_{1,1} = C_3(x_1, x_2, x_3)$ is a Y_1 type cycle, $Q_{1,2} = C_{12}(x_1, x_4, x_7, x_{10}, x_2, x_5, x_8, x_{11}, x_3, x_6, x_9, x_{12})$ and $Q_{1,3} = C_4(x_1, x_4, x_7, x_{10})$ are Y_2 type cycles and $Q_{1,4} = C_4(x_1, x_4, x_7, x_{10})$ is a Y_4 type cycle. In $M(3,4,1)$, $Q_{2,1} = C_3(y_1, y_2, y_3)$ is a Y_1 type cycle, $Q_{2,2} = C_{12}(y_1, y_4, y_7, y_{10}, y_3, y_6, y_9, y_{12}, y_2, y_5, y_8, y_{11})$ and $Q_{2,3} = C_{12}(y_1, y_4, y_7, y_{10}, y_2, y_5, y_8, y_{11}, y_3, y_6, y_9, y_{12})$ are Y_2 type cycles and $Q_{2,4} = C_5(y_2, y_5, y_8, y_{11}, y_1)$ is a Y_4 type cycle. In $M(3,4,2)$, $Q_{3,1} = C_3(z_1, z_2, z_3)$ is a Y_1 type cycle, $Q_{3,2} = C_4(z_1, z_4, z_7, z_{10})$ and $Q_{3,3} = C_{12}(z_1, z_4, z_7, z_{10}, z_3, z_6, z_9, z_{12}, z_2, z_5, z_8, z_{11})$ are Y_2 type cycles and $Q_{3,4} = C_4(z_3, z_6, z_9, z_{12})$ is a Y_4 type cycle.

In $M(i,j,k)$, observe that type Y_1 cycles have the same length and type Y_2 cycles have at most two different lengths. Since length$(Q_{2,4}) \ne$ length$(Q_{r,4})$ for $r = 1, 3$, $M(3,4,1) \ncong M(3,4,0)$, $M(3,4,2)$. Observe that, length $(Q_{1,1})$ = length$(Q_{3,1})$, { length$(Q_{1,2})$, length$(Q_{1,3})$} = { length $(Q_{3,2})$, length $(Q_{3,3})$} and length$(Q_{1,4})$ =length$(Q_{3,4})$. Now identifying $M(3,4,0)$ along the vertical boundray and cutting along the path $P(x_1, x_4, x_{17}, x_{10}, x_2)$ leads to Figure 3.4.6, i.e., $M(3,4,2)$. By the isomorphism map define in Lemma 3.4.2, $M(3,4,0) \cong M(3,4,2)$. Therefore, there are two DSEMs, up to isomorphism, of type $[3^3.4^2 : 4^4]_2$ with 12 vertices on the torus.

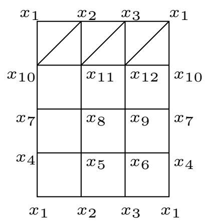

Figure 3.4.3: $M(3,4,0)$

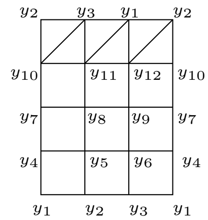

Figure 3.4.4: $M(3,4,1)$

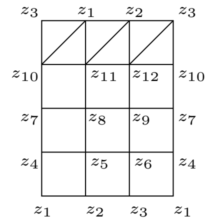

Figure 3.4.5: $M(3,4,2)$

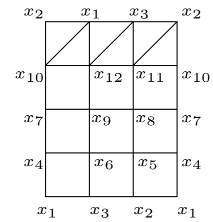

Figure 3.4.6: $M(3,4,2)$

Table 3.4 : DSEMs of type $[3^3.4^2 : 4^4]_2$ on the torus for $|V(M)| \leq 24$

$\|V(M)\|$	Isomorphic classes	Length of cycles	No of maps
12	$M(3,4,0)$, $M(3,4,2)$	$(3,\{4,12\},4)$	2
	$M(3,4,1)$	$(3,\{12,12\},5)$	
16	$M(4,4,0)$, $M(4,3,3)$	$(4,\{4,16\},4)$	2
	$M(4,4,1)$, $M(4,4,2)$	$(4,\{8,16\},5)$	
20	$M(5,4,0)$, $M(5,4,4)$	$(5,\{4,20\},4)$	3
	$M(5,4,1)$, $M(5,4,3)$	$(5,\{20,20\},5)$	
	$M(5,4,2)$	$(5,\{20,20\},6)$	
24	$M(6,4,0)$, $M(6,4,5)$	$(6,\{4,24\},4)$	5
	$M(6,4,1)$, $M(6,4,4)$	$(6,\{12,24\},5)$	
	$M(6,4,2)$, $M(6,4,3)$	$(6,\{8,12\},6)$	
	$M(3,8,0)$, $M(3,8,1)$	$(3,\{8,24\},8)$	
	$M(3,8,2)$	$(3,\{24,24\},10)$	

3.5 DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_1$

Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_1$ with the vertex set $V(M)$. For the existence of M we have $2|V_{(3^3,4^2)}| = |V_{(3^2,4,3,4)}|$ and $|V_{(3^2,4,3,4)}| = 4k$, for some $k \in \mathbb{N}$, see [11]. In this type DSEM, we consider a path of type, say W_1, as $P_1 = P(\ldots, w_i, z_j, z_{j+1}, z_{j+2}, z_{j+3}, w_{i+1}, \ldots)$ through a vertex, see in Figure 3.5.2, where the vertices denoted by $w_i's$ and $z_j's$ have face-sequences $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$ respectively. Observe that through a vertex v with face-sequence $(3^2, 4, 3, 4)$, we have two paths of type W_1, as shown in Figure 3.5.1.

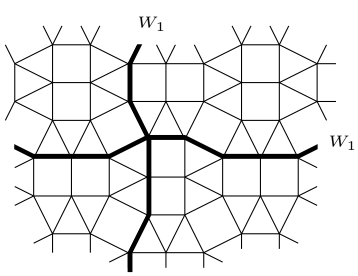

Figure 3.5.1: Paths of type W_1

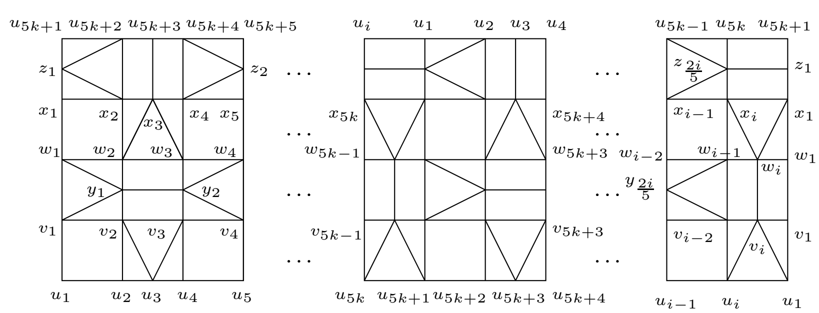

Figure 3.5.3: $M(i,4,5k)$

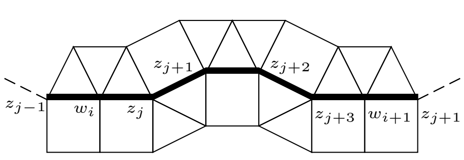

Figure 3.5.2: Path of type W_1 (indicated by bold line)

Consider a maximal path $P(v_1, v_2, \ldots, v_i)$ of the type W_1. By Lemmas 3.1.1, 3.1.3, there is an edge v_i-v_1 in M such that $P(v_1, v_2, \ldots, v_i) \cup \{v_i\text{-}v_1\}$ is a non-contractible cycle $Q = C_i(v_1, v_2, \ldots, v_i)$ of type W_1. Then as in Section 3.1, every DSEM of type $[3^3.4^2 : 3^2.4.3.4]_1$ has an $M(i,j,k)$ representation for some i, j, k. Figure 3.5.3 is an example of $M(i,4,5k)$ representation.

Lemma 3.5.1 *The DSEM M of type $[3^3.4^2 : 3^2.4.3.4]_1$ admits an $M(i,j,k)$ representation iff the following holds: (i) $i = 5m$, $m \in \mathbb{N}$ and j is even, (ii) number of vertices of $M(i,j,k) = 6ij/5 \geq 12$, (iii) if $j = 4m+2$, $m \in \mathbb{N} \cup \{0\}$ then $k \in \{5r+3 : 0 \leq r < i/5\}$, and if $j = 4m$, $m \in \mathbb{N}$ then $k \in \{5r : 0 \leq r < i/5\}$.*

Proof. Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_1$. An $M(i,j,k)$ of M has j number of W_1 type disjoint horizontal cycles of length i. Let $Q(1,0) = C_i(y_{0,0}, y_{0,1}, \ldots, y_{0,i-1})$, $Q(1,1) = C_i(y_{1,0}, y_{1,1}, \ldots, y_{1,i-1})$, $\ldots$, $Q(1, j-1) = C_i(y_{j-1,0}, y_{j-1,1}, \ldots, y_{j-1,i-1})$ denote horizontal cycles of type W_1. Note that, the number of vertices with face-sequence $(3^3, 4^2)$ lying between horizontal cycles $Q(1, (2s+1)(mod\, j))$ and $Q(1, (2s+2)(mod\, j))$, for $0 \leq s \leq j-1$, is $2i/5 \cdot j/2$. So, the total number of vertices in M is $n = ij + ij/5 = 6ij/5$. If $j = 1$, then $M(i,1,k)$ has no vertex with face-sequence $(3^3, 4^2)$ or $(3^2, 4, 3, 4)$. This is not possible. So $j \geq 2$. If $j \geq 2$ and j is not an even integer then some vertices in the base horizontal cycle do not follow the face-sequences $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$. So, j is even.

If j is even and $i < 5$ then the $M(4, j, k)$ representation has some vertices which do not follow the face-sequences $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$. So, $i \neq 4$. Similarly $i = 1, 2, 3$ is not possible. Thus $i \geq 5$. If $i \geq 5$ and not a multiple of 5, then $2|V(3^3, 4^2)| \neq |V(3^2, 4, 3, 4)|$. This is not possible. So $i = 5m$, where $m \in \mathbb{N}$ and $n = 6ij/5 \geq 12$.

If $j = 4m+2$, $m \in \mathbb{N} \cup \{0\}$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus \{5r+3 : 0 \leq r < i/5\}$ then we get some vertices which do not follow the face-sequences $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$. So, $k \in \{5r+3 : 0 \leq r < i/5\}$ for $j = 4m+2$, $m \in \mathbb{N} \cup \{0\}$. Proceeding similarly we see that if $j = 4m$, $m \in \mathbb{N}$ then $k \in \{5r : 0 \leq r < i/5\}$. This completes the proof. $\square$

Let $M(i,j,k)$ be a representation of a DSEM M of type $[3^3.4^2 : 3^2.4.3.4]_1$. Let $Q_{lh} = C_i(y_1, y_2, \ldots, y_i)$ and $Q_{uh} = C_i(y_{k+1}, y_{k+2}, \ldots, y_k)$ be the lower and upper horizontal cycles in the representation respectively. Let $P_1 = P(y_{k+1}, \ldots, y_{k_1})$ be a path through y_{k+1} of type W_1 that is not a part of horizontal cycles. Consider the paths $P_1' = P(y_{k+1}, \ldots, y_{k_1})$ and $P_1'' = P(y_{k_1}, \ldots, y_{k+1})$ in Q_{uh} such that $Q_{uh} = P_1' \cup P_1''$. Let $Q_{3,1} = P_1 \cup P_1'$ and $Q_{3,2} = P_1 \cup P_1''$. Define a cycle Q_3 of new type as

$$Q_3 = \begin{cases} Q_{3,1}, & \text{if length}(Q_{3,1}) \leq \text{length}(Q_{3,2}) \\ Q_{3,2}, & \text{if length}(Q_{3,1}) > \text{length}(Q_{3,2}). \end{cases} \tag{2}$$

We say that Q_3 is of type W_2. So we have cycles of types W_1, W_2 in $M(i,j,k)$.

For $t \in \{1,2\}$, let M_t be DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_1$ on n_t number of vertices with $n_1 = n_2$. Let $M_t(i_t, j_t, k_t)$ be a representation of M_t. Following equation (2), define cycles of type W_2 in R which is obtained by identifying $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(y_{0,0}, y_{1,0}, \ldots, y_{j_2-1,0}, y_{0,k_2})$ and then cutting along the path $P(y_{0,r}, y_{1,r-1}, y_{2,r-1}, y_{3,r}, \ldots, y_{0,r+k_2})$ or $P(y_{0,r}, y_{1,r-1}, y_{2,r-1}, y_{3,r}, \ldots, y_{0,r+k_2-1})$ for some $r = 5m+4$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \leq r \leq i_2 - 1$. In $M_t(i_t, j_t, k_t)$, let $Q_{t,\alpha}$, $\alpha = 1, 2$, denote non-homologous cycles of type W_1 and $Q_{t,3}$ denote cycles of type W_2. Let $l_{t,\alpha} = \text{length}(Q_{t,\alpha})$ for $\alpha = 1, 2, 3$. Then we show the following lemma.

Lemma 3.5.2 *The DSEMs $M_1 \cong M_2$ iff $(l_{1,1}, l_{1,2}, l_{1,3}) = (l_{2,r_1}, l_{2,r_2}, l_{2,3})$ for $r_1 \neq r_2 \in \{1,2\}$, where $l_{1,3}$ and $l_{2,3}$ are lengths of cycles of type W_2 in $M_1(i_1, j_1, k_1)$ and R respectively.*

Proof. Suppose that $(l_{1,1}, l_{1,2}, l_{1,3}) = (l_{2,r_1}, l_{2,r_2}, l_{2,3})$ for $r_1 \neq r_2 \in \{1,2\}$. This implies that $\{l_{11}, l_{12}\} = \{l_{21}, l_{22}\}$ and $l_{13} = l_{23}$. Then we make the following claim.

Claim. $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

By the definition, $M_t(i_t, j_t, k_t)$ has j_t number of W_1 type disjoint horizontal cycles of length i_t. Let $Q(1,0) = C_{i_1}(y_{0,0}, y_{0,1}, \ldots, y_{0,i_1-1}), Q(1,1) = C_{i_1}(y_{1,0}, y_{1,1}, \ldots, y_{1,i_1-1}), \ldots, Q(1, j_1-1) = C_{i_1}(y_{j_1-1,0}, y_{j_1-1,1}, \ldots, y_{j_1-1,i_1-1})$ denote W_1 type horizontal cycles in $M_1(i_1, j_1, k_1)$. Let $G_1(1, (2s+1)(mod\, j_1)) = \{x_{(2s+1)(mod\, j_1),0}, x_{(2s+1)(mod\, j_1),1}, \ldots, x_{(2s+1)(mod\, j_1),(2i_1-5)/5}\}$ be the set of vertices which lie between horizontal cycles $Q(1,(2s+1)(mod\, j_1))$ and $Q(1,(2s+2)(mod\, j_1))$ for $0 \le s \le j_1-1$. Similarly, let $Q(2,0) = C_{i_2}(z_{0,0}, z_{0,1}, \dots, z_{0,i_2-1}), Q(2,1) = C_{i_2}(z_{1,0}, z_{1,1}, \dots, z_{1,i_2-1}), \dots,$ $Q(2, j_2-1) = C_{i_2}(z_{j_2-1,0}, z_{j_2-1,1}, \dots, z_{j_2-1,i_2-1})$ denote W_1 type horizontal cycles in $M_2(i_2, j_2,\, k_2)$ and $G_2(2,(2s+1)(mod\, j_2)) = \{w_{(2s+1)(mod\, j_2),0}, w_{(2s+1)(mod\, j_2),1}, \dots, w_{(2s+1)(mod\, j_2),(2i_2-5)/5}\}$ be the set of vertices which lie between horizontal cycles $Q(1,(2s+1)(mod\, j_2))$ and $Q(1,(2s+2)(mod\, j_2))$ for $0 \le s \le j_2-1$. Now, we have the following cases.

Case 1: If $(i_1, j_1, k_1) = (i_2, j_2, k_2)$ then $i_1 = i_2,\ j_1 = j_2,\ k_1 = k_2$. Define an isomorphism $f : V(M_1(i_1, j_1, k_1)) \to V(M_2(i_2, j_2, k_2))$ such that $f(y_{g,h}) = z_{g,h}$ for $0 \le g \le j_1-1$, $0 \le h \le i_1-1$ and $f(x_{(2s+1)(mod\, j_1),h}) = w_{(2s+1)(mod\, j_1),h}$ for the vertices of $G_1(1,(2s+1)(mod\, j_1))$ and $G_2(2,(2s+1)(mod\, j_1))$ for all $0 \le s \le j_1-1$, $0 \le h \le (2i_1-5)/5$. By f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 2(a): If $i_1 = i_2$, $j_1 = j_2 = 4m$ and $k_1 \ne k_2$, where $m \in \mathbb{N}$, then identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, \dots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,r}, z_{1,r-1}, z_{2,r-1}, z_{3,r},$ $x_{3,\lfloor 2r/5 \rfloor}, z_{4,r}, \dots, z_{0,r+k_2})$ for some $r = 5m+4$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \le r \le i_2-1$. This gives representation R of $M_2(i_2, j_2, k_2)$ with a map $f' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f'(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2)}, & \text{if } 0 \le h \le i_2-1 \text{ and } 0 \le g \le j_2-1,\ g = 4m-1, 4m; \text{ where} \\ & m \in \mathbb{N} \cup \{0\} \\ z_{g,(i_2+r-h-1)(mod\, i_2)}, & \text{if } 0 \le h \le i_2-1 \text{ and } 0 \le g \le j_2-1,\ g \ne 4m-1, 4m; \text{ where} \\ & m \in \mathbb{N} \cup \{0\} \end{cases}$$

$f'(w_{(2s+1)(mod\, j_2),h}) = w_{(2s+1)(mod\, j_2),(2i_2/5+\lfloor 2r/5 \rfloor-h)(mod\, 2i_2/5)}$ for $0 \le s \le j_2-1$ and $0 \le h \le (2i_2-5)/5$.

In R, the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \dots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \dots,$ $z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2}, z_{0,r+k_2-1}, \dots, z_{0,r+k_2+1})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \dots,$ $z_{0,r+k_2})$ in Q' has length $r+i_2-(r+k_2) = i_2-k_2$. Note that, R has j_2 number of horizontal cycles of length i_2. So $R = M_2(i_2, j_2, i_2-k_2)$. Since $l_{1,3} = l_{2,3}$, length$(Q_{1,3})$ = length$(Q_{2,3})$. This implies $\min\{j_1+j_1/4+k_1, j_1+j_1/4+(i_1-k_1)\} = \min\{j_2+j_2/4+i_2-k_2, j_2+j_2/4+i_2-(i_2-k_2)\}$. Since $i_1 = i_2, j_1 = j_2$ and $k_1 \ne k_2$, it follows that $k_1 + 5j_1/4 \ne k_2 + 5j_2/4$ and $j_1 + j_1/4 + (i_1-k_1) \ne j_2 + j_2/4 + i_2 - (i_2-k_2)$. This gives $k_1 + 5j_1/4 = i_2 + 5j_2/4 - k_2 = i_2 + 5j_1/4 - k_2$ as $j_1 = j_2$. That is, $k_1 = i_2 - k_2$. Thus, $M_2(i_2, j_2, i_2-k_2) = M_1(i_1, j_1, k_1)$. So by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

(b): If $i_1 = i_2$, $j_1 = j_2 = 4m+2$ and $k_1 \ne k_2$, where $m \in \mathbb{N}$, identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, \dots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,r}, z_{1,r-1}, z_{2,r-1}, z_{3,r},$ $x_{3,\lfloor 2r/5 \rfloor}, z_{4,r}, \dots, z_{0,r+k_2-1})$ for some $r = 5m+4$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \le r \le i_2-1$. This gives representation R of M_2 with a map $f'' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f''(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2)}, & \text{if } 0 \le h \le i_2-1 \text{ and } 0 \le g \le s_2-1,\ g = 4m-1, 4m; \text{ where} \\ & m \in \mathbb{N} \cup \{0\} \\ z_{g,(i_2+r-h-1)(mod\, i_2)}, & \text{if } 0 \le h \le i_2-1 \text{ and } 0 \le g \le s_2-1,\ g \ne 4m-1, 4m; \text{ where} \\ & m \in \mathbb{N} \cup \{0\} \end{cases}$$

$f''(w_{(2s+1)(mod\, j_2),h}) = w_{(2s+1)(mod\, j_2),(2i_2/5+\lfloor 2r/5 \rfloor-h)(mod\, 2i_2/5))}$ for $0 \le s \le j_2-1$ and $0 \le h \le (2i_2-5)/5$.

In R, the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \dots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \dots,$ $z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2-1}, z_{0,r+k_2-2}, \dots, z_{0,r+k_2})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \dots,$ $z_{0,r+k_2-1})$ in Q' has length $r+i_2-(r+k_2-1) = i_2-k_2+1$. Note that, R has j_2 number of horizontal cycles of length i_2. So $R = M_2(i_2, j_2, i_2-k_2+1)$. By assumption, $l_{1,3} = l_{2,3}$, length$(Q_{1,3})$ = length$(Q_{2,3})$. This implies that $\min\{j_1 + \lfloor j_1/3 \rfloor + (k_1-1), j_1 + \lfloor j_1/3 \rfloor + (i_1-k_1+1)\}$ $= \min\{j_2 + \lfloor j_2/3 \rfloor + (i_2-k_2+1)-1, j_2 + \lfloor j_2/3 \rfloor + i_2 - (i_2-k_2)\}$. If $k_1 + j_1 + \lfloor j_1/3 \rfloor - 1 = k_2 + j_2 + \lfloor j_2/3 \rfloor$, it implies $k_1 - k_2 = 1$ since $j_1 = j_2$. This is not possible since $k_1 - k_2 = 5m$, where $m \in \mathbb{N}$. This gives that $k_1 + j_1 + \lfloor j_1/3 \rfloor - 1 = i_2 + j_2 + \lfloor j_2/3 \rfloor - k_2 = i_2 + j_1 + \lfloor j_1/3 \rfloor - k_2$ as $j_1 = j_2$. That is, $k_1 = i_2 - k_2 + 1$. Thus $M_2(i_2, j_2, i_2 - k_2 + 1) = M_1(i_1, j_1, k_1)$. So by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

(c): If $i_1 = i_2$, $j_1 = j_2 = 2$ and $k_1 \neq k_2$, identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, z_{0,k_2})$ of $M_2(i_2, j_2, k_2)$ and then cut along the path $P(z_{0,r}, z_{1,r-1}, z_{0,r+k_2-1})$ for some $r = 5m + 4$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \leq r \leq i_2 - 1$. This leads to representation R of M_2 with a map $f''' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f'''(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2)}, & \text{if } g = 0 \text{ and } 0 \leq h \leq i_2 - 1 \\ z_{g,(i_2+r-h-1)(mod\, i_2)}, & \text{if } g = 1 \text{ and } 0 \leq h \leq i_2 - 1 \end{cases}$$

$f'''(w_{1,h}) = w_{1,(2i_2/5+\lfloor 2r/5 \rfloor - h)(mod\, 2i_2/5))}$ for $0 \leq h \leq (2i_2 - 5)/5$.

In R, the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \ldots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \ldots, z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2-1}, z_{0,r+k_2-2}, \ldots, z_{0,r+k_2})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \ldots, z_{0,r+k_2-1})$ in Q' has length $r + i_2 - (r + k_2 - 1) = i_2 - k_2 + 1$. In this process, R has j_2 number of horizontal cycles of length i_2. Thus, we have $R = M_2(i_2, j_2, i_2 - k_2 + 1)$. By assumption, $l_{1,3} = l_{2,3}$, length$(Q_{1,3})$ = length$(Q_{2,3})$. This implies $\min\{j_1 + j_1/2 + (k_1 - 1), j_1 + j_1/2 + (i_1 - k_1 + 1)\} = \min\{j_2 + j_2/2 + (i_2 - k_2 + 1) - 1, j_2 + j_2/2 + i_2 - (i_2 - k_2)\}$. If $k_1 + 3j_1/2 - 1 = k_2 + 3j_2/2$ then $k_1 - k_2 = 1$, as $j_1 = j_2$. This is not possible since $k_1 - k_2 = 5m$, for $m \in \mathbb{N}$. Therefore $k_1 + 3j_1/2 - 1 = i_2 + 3j_2/2 - k_2 = i_2 + 3j_1/2 - k_2$, as $j_1 = j_2$. That is, $k_1 = i_2 - k_2 + 1$. Thus, $(i_2, j_2, i_2 - k_2 + 1) = (i_1, j_1, k_1)$. Hence by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 3: If $i_1 \neq i_2$, identify boundaries of $M_2(i_2, j_2, k_2)$ and cut M_2 along a cycle of type W_1 of length i_1 to get another representation say $R' = M_2(i_3, j_3, k_3)$ of M_2. This implies that $i_1 = i_3$ and $j_1 = j_3$ as $n_1 = 6i_1j_1/5 = 6i_3j_3/5 = n_2$. Thus $R' = M_2(i_1, j_1, k_3)$. If $k_1 \neq k_3$ then we are in Case 2. If $k_1 = k_3$ then $M_1 \cong M_2$ by f in Case 1. This completes the claim. Therefore, $M_1 \cong M_2$.

Conversely, let $M_1 \cong M_2$ by an isomorphism f. Let $Q_{1,\alpha}$ and $Q_{2,\alpha}$, $\alpha = 1, 2$, denote non-homologous cycles of type W_1 in M_1 and M_2 respectively. Also, let $Q_{1,3}$ and $Q_{2,3}$ denote cycles of type W_2 in M_1 and R respectively. Let $f : V(M_1) \to V(M_2)$ be such that $f(Q_{1,\alpha}) = Q_{2,\alpha}$ for $\alpha = 1, 2, 3$. Since f is an isomorphism, length$(Q_{1,\alpha})$ = length$(f(Q_{1,\alpha}))$ = length$(Q_{2,\alpha})$. So, $\{l_{11}, l_{12}\} = \{l_{21}, l_{22}\}$ and $l_{13} = l_{23}$. Hence, M_1 and M_2 have the same cycle-type. $\square$

Now computing DSEMs for the first four admissible values of $|V(M)|$, we get Table 3.5. We illustrate this computation for $|V(M)| = 24$ as follows.

Example 3.5.1 Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_1$ with 24 vertices on the torus. By Lemma 3.5.1, M has three $M(i, j, k)$ representation, namely, $M(5, 4, 0)$, $M(10, 2, 3)$ and $M(10, 2, 8)$, see Figures 3.5.4, 3.5.5, 3.5.6 respectively. In $M(5, 4, 0)$, $Q_{1,1} = C_5(u_1, u_2, u_3, u_4, u_5)$ and $Q_{1,2} = C_5(u_1, v_1, w_1, x_1, z_1)$ are W_1 type cycles and $Q_{1,3} = C_5(u_1, v_1, w_1, x_1, z_1)$ is a W_2 type cycle. In $M(10, 2, 3)$, $Q_{2,1} = C_{10}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})$ and $Q_{2,2} = C_{10}(u_1, v_1, u_4, v_3, y_2, u_6, v_6, u_9, v_8, y_4)$ are W_1 type cycles, and $Q_{2,3} = C_5(u_4, v_3, y_2, u_6, u_5)$ is a W_2 type cycle. In $M(10, 2, 8)$, $Q_{3,1} = C_{10}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})$ and $Q_{3,2} = C_{10}(u_1, v_1, u_9, v_8, y_4, u_6, v_6, u_4, v_3, y_2)$ are W_1 type cycles, and $Q_{3,3} = C_6(u_9, v_8, y_4, u_6, u_7, u_8)$ is a W_2 type cycle.

Since $\{$length$(Q_{1,1})$, length$(Q_{1,2})\} \neq \{$length$(Q_{r,1})$, length$(Q_{r,2})\}$ for $r = 2, 3$, $M(5, 4, 0) \not\cong M(10, 2, 3), M(10, 2, 8)$. Now identifying $M(10, 2, 3)$ along the path $P(u_1, v_1, u_4)$ and cutting along the path $P(u_5, v_4, u_7)$ leads to Figure 3.5.6, i.e., $M(10, 2, 8)$. By Lemma 3.5.2, $M(10, 2, 3) \cong M(10, 2, 8)$. Therefore, there are two DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_1$ with 24 vertices.

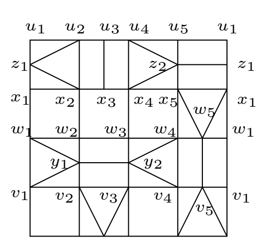

Figure 3.5.4: $M(5, 4, 0)$

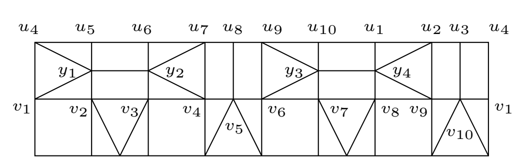

Figure 3.5.5: $M(10, 2, 3)$

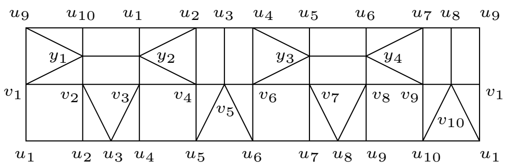

Figure 3.5.6: $M(10, 2, 8)$

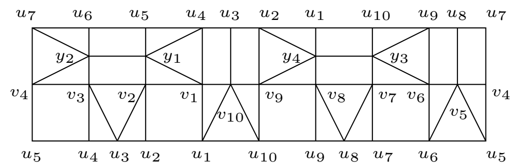

Figure 3.5.7: $M(10, 2, 8)$

Table 3.5 : DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_1$ on the torus for $|V(M)| \leq 48$

$\lvert V(M)\rvert$	Isomorphic classes	Length of cycles	No of maps
12	$M(5, 2, 3)$	$(\{5, 5\}, 5)$	1
24	$M(5, 4, 0)$	$(\{5, 5\}, 5)$	2
	$M(10, 2, 3), M(10, 2, 8)$	$(\{10, 10\}, 5)$	
36	$M(5, 6, 3), M(15, 2, 8)$	$(\{5, 15\}, 10)$	2
	$M(15, 2, 3), M(15, 2, 13)$	$(\{15, 15\}, 5)$	
48	$M(5, 8, 0), M(10, 4, 0)$	$(\{5, 10\}, 10)$	4
	$M(10, 4, 5)$	$(\{10, 10\}, 10)$	
	$M(20, 2, 3), M(20, 2, 18)$	$(\{20, 20\}, 5)$	
	$M(20, 2, 8), M(20, 2, 13)$	$(\{20, 20\}, 10)$	

3.6 DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_2$

Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_2$ with vertex set $V(M)$. Then $|V_{(3^3,4^2)}| = |V_{(3^2,4,3,4)}| = 2k$, for some $k \in \mathbb{N}$, see [11], where $|V_{(3^3,4^2)}|$ and $|V_{(3^2,4,3,4)}|$ are same as in Section 3.5. We consider following two types of paths in M as follows.

A path $P_1 = P(\ldots, u_i, v_j, u_{i+1}, v_{j+1}, \ldots)$ in the underlying graph of M, say of type Z_1, see in Figure 3.6.1, where the vertices $u_i's$ and $v_j's$ have face-sequences $(3^2, 4, 3, 4)$ and $(3^3, 4^2)$ respectively.

A path $P_2 = P(\ldots, u_{i-1}, u_i, u_{i+1}, \ldots)$ in the underlying graph of M, say of type Z_2, see in Figure 3.6.1, where every vertex of the path has face-sequence either $(3^3, 4^2)$ or $(3^2, 4, 3, 4)$.

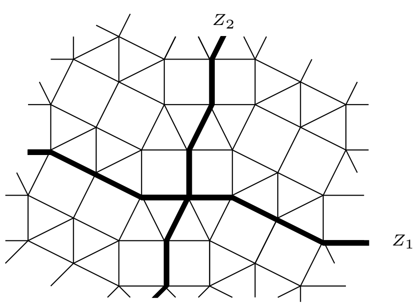

Figure 3.6.1: Paths of types Z_1 and Z_2

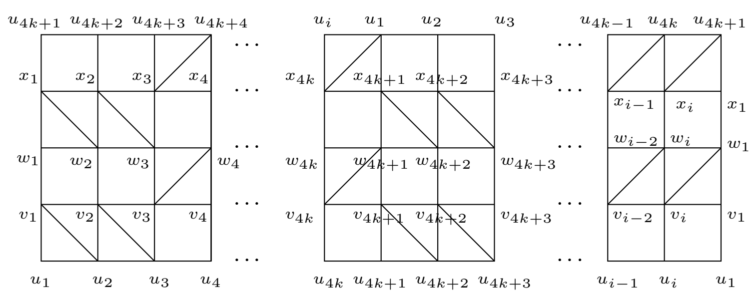

Figure 3.6.2: $M(i, 4, 4k)$

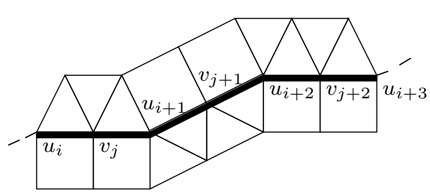

Figure 3.6.3: Path of type Z_1 (indicated by bold line)

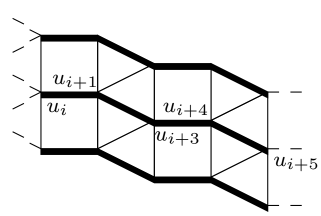

Figure 3.6.4: Paths of type Z_2 (indicated by bold lines)

Consider a maximal path $P(v_1, v_2, \ldots, v_i)$ of the type Z_α, $\alpha \in \{1, 2\}$. By Lemmas 3.1.1, 3.1.3, there is an edge v_i-v_1 in M such that $P(v_1, v_2, \ldots, v_i) \cup \{v_i\text{-}v_1\}$ is a non-contractible cycle $Q = C_i(v_1, v_2, \ldots, v_i)$. Let $u \in V(M)$ and Q_α be cycles of type Z_α through u. As in Section 3.1, define an $M(i,j,k)$ representation of M for some i,j,k by cutting M along the cycle Q_1 and then cutting it along the cycle Q_2, for example see $M(i,4,4k)$ in Figure 3.6.2.

Lemma 3.6.1 *A DSEM M of type $[3^3.4^2 : 3^2.4.3.4]_2$ admits an $M(i,j,k)$ representation iff the following holds: (i) $j \geq 2$ and j even, (ii) if $j = 2$ then $i \geq 8$, and if $j \geq 4$ then $i \geq 4$, also $i = 4m$, where $m \in \mathbb{N}$, (iii) $ij \geq 16$, (iv) if $j = 2$ then $k \in \{4r : 0 < r < i/4\}$, and if $j \geq 4$ then $k \in \{4r : 0 \leq r < i/4\}$.*

Proof. Let M be a DSEM of the above type with n vertices. Then $M(i,j,k)$ has j number of Z_1 type disjoint horizontal cycles of length i. Since all the vertices of M lie in these cycles, the number of vertices in M is $n = ij$. If $j = 1$, then $M(i,1,k)$ representation has no vertex having face-sequence $(3^3,4^2)$ or $(3^2,4,3,4)$, this can not be true. So $j \geq 2$. If $j \geq 2$ and j is not an even integer then after identifying the boundaries of $M(i,j,k)$, we get some vertices in the base horizontal cycle which do not follow the face-sequences $(3^3,4^2)$ and $(3^2,4,3,4)$. So, j is even.

If $j = 2$ and $i < 8$ then the $M(7,2,k)$ has some vertices which do not follow the face-sequences $V_{(3^3,4^2)}$ and $V_{(3^2,4,3,4)}$. So, $i \neq 7$. Similarly $i = 1,2,3,4,5,6$ is not possible. Thus $i \geq 8$ for $j = 2$.

If $j \geq 4$ and $i < 4$ then as above we get that $i \neq 1,2,3$. So $i \geq 4$. If $i \geq 4$ and is not a multiple of 4, then $|V(3^3,4^2)| \neq |V(3^2,4,3,4)|$. A contradiction. So, $i = 4m$, where $m \in \mathbb{N}$ and $n = ij \geq 16$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus \{4r : 0 < r < i/4\}$ then we get some vertices which do not follow the face-sequences $(3^3,4^2)$ and $(3^2,4,3,4)$. So, $k \in \{4r : 0 < r < i/4\}$ for $j = 2$. Proceeding similarly, we see that if $j \geq 4$ then $k \in \{4r : 0 \leq r < i/4\}$. This completes the proof. □

Let $M(i,j,k)$ be a representation of a DSEM M of the type $[3^3.4^2 : 3^2.4.3.4]_2$. Let $Q_{lh} = C_i(x_1,x_2,\ldots,x_i)$ and $Q_{uh} = C_i(x_{k+1},x_{k+2},\ldots,x_k)$ be the lower and upper horizontal cycles in $M(i,j,k)$ respectively. Let $P_1 = P(x_{k+1},\ldots,x_{k_1})$ be a path of type Z_2 through x_{k+1}. Consider the paths $P_1' = P(x_{k+1},\ldots,x_{k_1})$ and $P_1'' = P(x_{k_1},\ldots,x_{k+1})$ in Q_{uh} such that $Q_{uh} = P_1' \cup P_1''$. Let $Q_{3,1} = P_1 \cup P_1'$ and $Q_{3,2} = P_1 \cup P_1''$. Now define a new cycle Q_3 as

$$Q_3 = \begin{cases} Q_{3,1}, & \text{if length}(Q_{3,1}) \leq \text{length}(Q_{3,2}) \\ Q_{3,2}, & \text{if length}(Q_{3,1}) > \text{length}(Q_{3,2}). \end{cases} \tag{3}$$

We say that Q_3 is of type Z_3. Thus, we have cycles of three types Z_1, Z_2 and Z_3 in $M(i,j,k)$.

For $t \in \{1,2\}$, let M_t be two DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_2$ with the number of vertices n_t such that $n_1 = n_2$. Let $M_t(i_t,j_t,k_t)$ be a representation of M_t and $Q_{t,\alpha}$ be the cycle of type Z_α, $\alpha = 1,2,3$. If $l_{t,\alpha}$ = length of the cycle of type Z_α in $M_t(i_t,j_t,k_t)$ then we say that $M_t(i_t,j_t,k_t)$ has cycle-type $(l_{t,1},l_{t,2},l_{t,3})$. Now, we show the following lemma.

Lemma 3.6.2 *The DSEMs $M_1 \cong M_2$ if and only if they have same cycle-type.*

Proof. Let M_1 and M_2 be two DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_2$ with the same number of vertices such that they have same cycle-type. This implies that $l_{1,1} = l_{2,1}$, $l_{1,2} = l_{2,2}$ and $l_{1,3} = l_{2,3}$

Claim. $M_1(i_1,j_1,k_1) \cong M_2(i_2,j_2,k_2)$.

By the definition, $M_t(i_t,j_t,k_t)$ has j_t number of Z_1 type horizontal cycles, say $Q_0 = C_{i_1}(x_{0,0},x_{0,1},\ldots,x_{0,i_1-1})$, $Q_1 = C_{i_1}(x_{1,0},x_{1,1},\ldots,x_{1,i_1-1}),\ldots,Q_{j_1-1} = C_{i_1}(x_{j_1-1,0},x_{j_1-1,1},\ldots,x_{j_1-1,i_1-1})$ in $M_1(i_1,j_1,k_1)$ and $Q_0' = C_{i_2}(y_{0,0},y_{0,1},\ldots,y_{0,i_2-1})$, $Q_1' = C_{i_2}(y_{1,0},y_{1,1},\ldots,y_{1,i_2-1}),\ldots,Q_{j_2-1}' = C_{i_2}(y_{j_2-1,0},y_{j_2-1,1},\ldots,y_{j_2-1,i_2-1})$ in $M_2(i_2,j_2,k_2)$. Now, we have the following cases.

Case 1: If $(i_1,j_1,k_1) = (i_2,j_2,k_2)$ then $i_1 = i_2 = i$, $j_1 = j_2 = j$, $k_1 = k_2 = k$. Define an $f : V(M_1(i_1,j_1,k_1)) \to V(M_2(i_2,j_2,k_2))$ such that $f(x_{g,h}) = y_{g,h}$ for $0 \leq g \leq j-1$ and $0 \leq h \leq i-1$. So by f, $M_1(i_1,j_1,k_1) \cong M_2(i_2,j_2,k_2)$.

Case 2: If $i_1 \neq i_2$, then it contradicting the fact $l_{1,1} = l_{2,1}$. Thus $i_1 = i_2$.

Case 3: If $j_1 \neq j_2$, then $n_1 = i_1j_1 \neq i_2j_2 = n_2$ as $i_1 = i_2$. A contradiction as $n_1 = n_2$. So, $j_1 = j_2$.
Case 4: If $i_1 = i_2$, $j_1 = j_2$ and $k_1 \neq k_2$. By $l_{1,3} = l_{2,3}$, we see length($Q_{1,3}$)=length($Q_{2,3}$). This implies $\min\{k_1 + j_1, j_1 + (i_1 - k_1)\}= \min\{k_2 + j_2, j_2 + (i_2 - k_2)\}$. Also, the condition follows that $k_1+j_1 \neq k_2+j_2$ and $j_1+(i_1-k_1) \neq j_2+(i_2-k_2)$. This gives that $k_1+j_1 = i_2+j_2-k_2 = i_1+j_1-k_2$ as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1-k_1$. Now identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(y_{0,0}, y_{1,0}, \ldots, y_{j_2-1,0}, y_{0,k_2})$ and then cut along the path $P(y_{0,r}, y_{1,r}, \ldots, y_{0,r+k_2})$ for some even $0 \leq r \leq i_2 - 1$, where two triangles and one quadrangle are incident at the vertex $y_{0,r}$. This gives another representation of M_2, say R, with a map $f' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that $f'(y_{g,h}) = y_{g,(i_2+r-h)(mod\, i_2))}$ for $0 \leq g \leq j_2 - 1$ and $0 \leq h \leq i_2 - 1$.

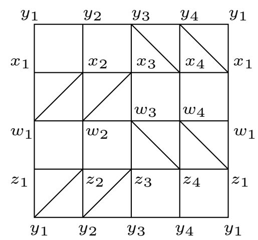

Figure 3.6.5: R

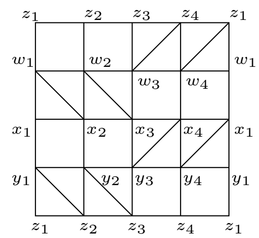

Figure 3.6.6: R'

Let $Q''_0, Q''_1, \ldots, Q''_{j_2-1}$ be horizontal cycles in R. In R, the lower and upper horizontal cycles are $Q_{i_2}(y_{0,r}, y_{0,r-1}, \ldots, y_{0,0}, y_{0,i_2-1}, \ldots, y_{0,r+1})$ and $Q_{i_2}(y_{0,r+k_2}, y_{0,r+k_2-1}, \ldots, y_{0,r+k_2+1})$ respectively. Observe that R does not fulfill the $M(i, j, k)$ condition because two triangles are incident at the beginning vertex $y_{0,r}$ in the base horizontal cycle $Q''_0 = C_{i_2}(y_{0,r}, y_{0,r-1}, \ldots, y_{0,i_2-1}, y_{0,0}, y_{0,1}, \ldots, y_{0,r+1})$. (For an example, R in Figure 3.6.5 is not of type $M(i, j, k)$ as their are two triangles incident at y_1). Now identify R along $Q''_0 = C_{i_2}(y_{0,r}, y_{0,r-1}, \ldots, y_{0,r+1})$ and then cut along $Q''_1 = C_{i_2}(y_{1,r}, y_{1,r-1}, \ldots, y_{1,r+1})$. This gives another representation say R' of $M_2(i_2, j_2, k_2)$, where $Q''_1 = C_{i_2}(y_{1,r}, y_{1,r-1}, \ldots, y_{1,r+1})$ denote the base horizontal cycle. (For an example, R' in Figure 3.6.6 is defined from R). This process defines a map $f'' : R \to R'$ such that $f''(Q''_s) = Q''_{(1-s)(mod\, j_2)}$ for $0 \leq s \leq j_2 - 1$. Thus, we redefine R to a new desired representation R' of M_2. In this processes, R' has j_2 number of Z_1 type horizontal cycles of length i_2, as we are changing the order of horizontal cycles. So, R' has well defined $M(i_2, j_2, i_2 - k_2)$ representation. Now $i_2 - k_2 = i_2 - (i_1 - k_1) = k_1$ since $i_1 = i_2$ and $k_2 = i_1 - k_1$. Thus, $M_2(i_2, j_2, i_2 - k_2) = M_1(i_1, j_1, k_1)$. Therefore by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$. So, the claim follows. Hence, $M_1 \cong M_2$.

The converse part follows from the converse part of the Lemma 3.1.7. Thus the proof. □

By Lemmas 3.6.1, 3.6.2, computing DSEMs for the first four admissible values of $|V(M)|$, we get Table 3.6. For $|V(M)| = 24$, the computations is illustrated as follows.

Example 3.6.1 Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_2$ with 24 vertices on the torus. By Lemma 3.6.1, M has three $M(i, j, k)$ representation, namely, $M(4, 6, 0), M(12, 2, 4)$ and $M(12, 2, 8)$, see Figures 3.6.7, 3.6.8, 3.6.9 respectively. In $M(4, 6, 0)$, $Q_{1,1} = C_4(u_1, u_2, u_3, u_4), Q_{1,2} = C_6(u_1, v_1, w_1, x_1, y_1, z_1)$, and $Q_{1,3} = C_6(u_1, v_1, w_1, x_1, y_1, z_1)$ are cycles of type Z_1, Z_2, Z_3 respectively. In $M(12, 2, 4)$, $Q_{2,1} = C_{12}(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$, $Q_{2,2} = C_6(y_1, z_1, y_5, z_5, y_9, z_9)$, and $Q_{2,3} = C_6(y_5, z_5, y_9, y_8, y_7, y_6)$ are cycles of type Z_1, Z_2, Z_3 respectively. In $M(12, 2, 8)$, $Q_{3,1} = C_{12}(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$, $Q_{3,2} = C_6(y_1, z_1, y_9, z_9, y_5, z_5)$, and $Q_{3,3} = C_6(y_9, z_9, y_5, y_6, u_7, u_8)$ are cycles of type Z_1, Z_2, Z_3 respectively.

Since length$(Q_{1,1}) \neq$ length$(Q_{r,1})$ for $r = 2, 3$, $M(4, 6, 0) \ncong M(12, 2, 4), M(12, 2, 8)$. We identify $M(12, 2, 4)$ along the path $P(y_1, z_1, y_5)$ and cut along the path $P(y_3, z_3, y_7)$ and next identify along horizontal cycle and cut along the cycle $C_{12}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12})$. This gives $M(12, 2, 8)$ represntation, given in Figure 3.6.10. By Lemma 3.6.2, $M(12, 2, 4) \cong M(12, 2, 8)$. Hence, there are two DSEMs, up to isomorphism, of type $[3^3.4^2 : 3^2.4.3.4]_2$ with 24 vertices on the torus.

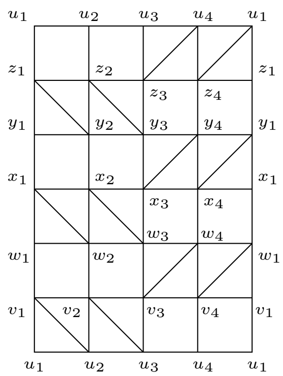

Figure 3.6.7: $M(4,6,0)$

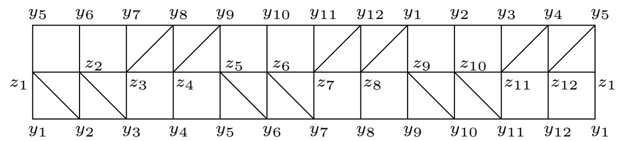

Figure 3.6.8: $M(12,2,4)$

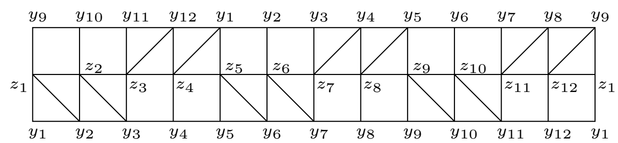

Figure 3.6.9: $M(12,2,8)$

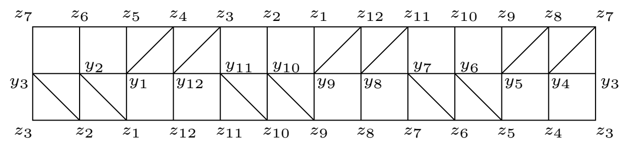

Figure 3.6.10: $M(12,2,8)$

Table 3.6 : DSEMs of type $[3^3.4^2 : 3^2.4.3.4]_2$ on the torus for $|V(M)| \leq 40$

$\lvert V(M) \rvert$	Isomorphic classes	Length of cycles	No of maps
16	$M(4,4,0)$	$(4,4,4)$	2
	$M(8,2,4)$	$(8,4,6)$	
24	$M(4,6,0)$	$(4,6,6)$	2
	$M(12,2,4), M(12,2,8)$	$(12,6,6)$	
32	$M(4,8,0)$	$(4,8,8)$	5
	$M(8,4,0)$	$(8,4,4)$	
	$M(8,4,4)$	$(8,8,8)$	
	$M(16,2,4), M(16,2,12)$	$(16,8,6)$	
	$M(16,2,8)$	$(16,4,10)$	
40	$M(4,10,0)$	$(4,10,10)$	3
	$M(20,2,4), M(20,2,16)$	$(20,10,6)$	
	$M(20,2,8), M(20,2,12)$	$(20,10,10)$	

3.7 DSEMs of type $[3^6 : 3^2.4.3.4]$

Let M be a DSEM of type $[3^3.4^2 : 3^2.4.3.4]_1$ with vertex set $V(M)$. Then $6|V_{(3^6)}| = |V_{(3^2,4,3,4)}|$, where $|V_{(3^6)}|$ and $|V_{(3^2,4,3,4)}|$ denote the cardinality of sets $V_{(3^6)}$ and $V_{(3^2,4,3,4)}$ respectively. Now consider following two types of paths in M as follows.

A path $P_1 = P(\ldots u_{i-1}, u_i, u_{i+1}, \ldots)$ in M, say of type H_1, where every vertex of the path has face-sequence $(3^2, 4, 3, 4)$, see in Figure 3.7.1.

A path $P_2 = P(\ldots, u_i, v_j, v_{j+1}, v_{j+2}, v_{j+3}, u_{i+1}, \ldots)$ in M, say of type H_2, where the vertices $u_i's$ and $v_j's$ have face-sequences (3^6) and $(3^2, 4, 3, 4)$ respectively, see in Figure 3.7.1.

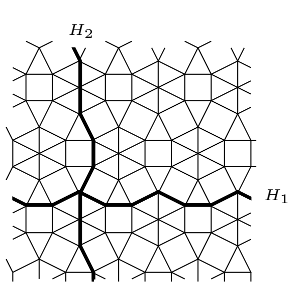

Figure 3.7.1: Paths of types H_1 and H_2

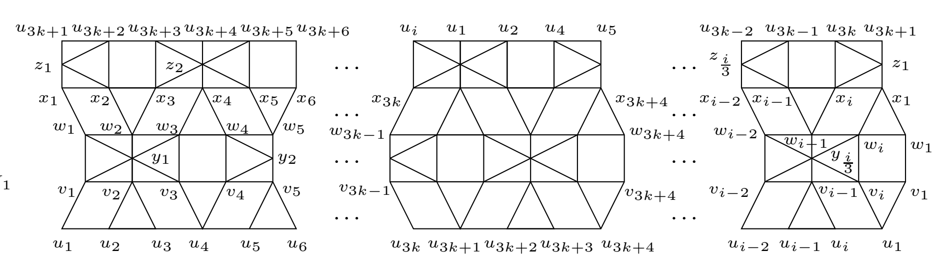

Figure 3.7.2: $M(i,4,3k)$

Consider a maximal path $P(v_1, v_2, \ldots, v_i)$ of the type H_α, $\alpha \in \{1, 2\}$. By Lemmas 3.1.1, 3.1.3, there is an edge v_i-v_1 in M such that $P(v_1, v_2, \ldots, v_i) \cup \{v_i\text{-}v_1\}$ is a non-contractible cycle $Q = C_i(v_1, v_2, \ldots, v_i)$. Let $u \in V(M)$ and Q_α be cycles of type H_α through u. As in Section 3.1, define an $M(i, j, k)$ representation for a DSEM M of type $[3^6 : 3^2.4.3.4]$ by cutting M first along the Q_1 cycle and then cutting along the Q_2 cycle. Figure 3.7.2 shows $M(i, 4, 3k)$.

Lemma 3.7.1 *A DSEM M of type $[3^6 : 3^2.4.3.4]$ admits an $M(i, j, k)$ representation iff the following holds: (i) $j \geq 2$ even, (ii) if $j = 2$ then $i \geq 9$ and if $j \geq 4$ then $i \geq 6$, also $i = 3m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i, j, k) = 7ij/6 \geq 21$, (iv) if $j = 2$ then $k \in \{3r + 2 : 0 < r < (i - 3)/3\}$, if $j = 4m + 2$ then $k \in \{3r + 2 : 0 \leq r < i/3\}$, and if $j = 4m$ then $k \in \{3r : 0 \leq r < i/3\}$, $m \in \mathbb{N}$.*

Proof. Let M be a DSEM of above type with n vertices. Then an $M(i, j, k)$ of M has j number of H_1 type disjoint horizontal cycles, say $Q(1, 0) = C_i(z_{0,0}, z_{0,1}, \ldots, z_{0,i-1}), Q(1, 1) = C_i(z_{1,0}, z_{1,1}, \ldots, z_{1,i-1}), \ldots, Q(1, j-1) = C_i(z_{j-1,0}, z_{j-1,1}, \ldots, z_{j-1,i-1})$. Observe that, the number of vertices with face-sequence (3^6) lying between horizontal cycles $Q(1, (2s+1)(mod\, j))$ and $Q(1, (2s+2)(mod\, j))$, for $0 \leq s \leq j-1$, is $i/3 \cdot j/2$. So, the total number of vertices in M is $n = ij + ij/6 = 7ij/6$. If $j = 1$ then $M(i, 1, k)$ has no vertex with face-sequence (3^6) or $(3^2, 4, 3, 4)$, which can not be true. So $j \geq 2$. If $j \geq 2$ and j is not an even integer then after identifying the boundaries, some vertices in the lower horizontal cycle does not follow the face-sequences (3^6) and $(3^2, 4, 3, 4)$. So j is even.

If $j = 2$ and $i < 9$ then $M(i, j, k)$ has some vertices which do not follow the face-sequences $V_{(3^6)}$ and $V_{(3^2,4,3,4)}$. Thus $i \geq 9$ for $j = 2$.

If $j \geq 4$ and $i < 6$ then as above we get that $i \neq 1, 2, 3, 4, 5$. Thus $i \geq 6$ for $j \geq 4$. If $i \geq 6$ and is not a multiple of 3, then $6|V(3^6)| \neq |V(3^2, 4, 3, 4)|$. So, $i = 3m$, where $m \in \mathbb{N}$ and $n = 7ij/6 \geq 21$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus \{3r + 2 : 0 < r < (i-3)/3\}$ then we get some vertices which do not follow the face-sequences (3^6) and $(3^2, 4, 3, 4)$. So, $k \in \{3r + 2 : 0 < r < (i-3)/3\}$ for $j = 2$. Proceeding similarly, we see that if $j = 4m$ then $k \in \{3r : 0 \leq r < i/3\}$ and if $j = 4m + 2$ then $k \in \{3r + 2 : 0 \leq r < i/3\}$, where $m \in \mathbb{N}$. This completes the proof. $\square$

Let $M(i, j, k)$ be a representation of a DSEM M of the type $[3^6 : 3^2.4.3.4]_1$. Let $Q_{lh} = C_i(y_1, y_2, \ldots, y_i)$ and $Q_{uh} = C_i(y_{k+1}, y_{k+2}, \ldots, y_k)$ be the lower and upper horizontal cycles in $M(i, j, k)$ respectively. Let $P_1 = P(y_{k+1}, \ldots, y_{k_1})$ be a path through y_{k+1} of type H_2. Consider the paths $P_1' = P(y_{k+1}, \ldots, y_{k_1})$ and $P_1'' = P(y_{k_1}, \ldots, y_{k+1})$ in Q_{uh} such that $Q_{uh} = P_1' \cup P_1''$. Let $Q_{3,1} = P_1 \cup P_1'$ and $Q_{3,2} = P_1 \cup P_1''$. Define a new cycle Q_3 as

$$Q_3 = \begin{cases} Q_{3,1}, & \text{if length}(Q_{3,1}) \leq \text{length}(Q_{3,2}) \\ Q_{3,2}, & \text{if length}(Q_{3,1}) > \text{length}(Q_{3,2}). \end{cases} \tag{4}$$

We say Q_3 of type H_3. So, we have cycles of types H_r, $r = 1, 2, 3$ in $M(i, j, k)$.

For $t \in \{1, 2\}$, let M_t be DSEMs of type $[3^6 : 3^2.4.3.4]$ on n_t number of vertices such that $n_1 = n_2$. Let $M_t(i_t, j_t, k_t)$ be representation of M_t. We define cycles of type H_3 in R which is obtained by identifying $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(y_{0,0}, y_{1,0}, \ldots, y_{j_2-1,0}, y_{0,k_2})$ and then cutting along the path $P(y_{0,r}, y_{1,r-1}, y_{2,r-1}, y_{3,r}, \ldots, y_{0,r+k_2})$ or $P(y_{0,r}, y_{1,r-1}, y_{2,r-1}, y_{3,r}, \ldots, y_{0,r+k_2-1})$ for some $r = 3m$, where $m \in \mathbb{N}$ and $0 \leq r \leq i_2 - 1$. Let $Q_{t,\alpha}$ be cycles of type H_α and $l_{t,\alpha}$ = length of cycles of type H_α for $\alpha = 1, 2, 3$. Then we have following lemma.

Lemma 3.7.2 *The DSEMs $M_1 \cong M_2$ iff $(l_{1,1}, l_{1,2}, l_{1,3}) = (l_{2,1}, l_{2,2}, l_{2,3})$, where $l_{1,3}$ and $l_{2,3}$ are lengths of cycles of type H_3 in $M_1(i_1, j_1, k_1)$ and R respectively.*

Proof. Let M_t, $t = 1, 2$, be two DSEMs of type $[3^6 : 3^2.4.3.4]$ with same number of vertices and $(l_{1,1}, l_{1,2}, l_{1,3}) = (l_{2,1}, l_{2,2}, l_{2,3})$. This implies that $l_{1,1} = l_{2,1}$, $l_{1,2} = l_{2,2}$ and $l_{1,3} = l_{2,3}$.

Claim. $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

By definition, $M_t(i_t, j_t, k_t)$ has j_t number of H_1 type disjoint horizontal cycles of length i_t. Let $Q(1,0) = C_{i_1}(y_{0,0}, y_{0,1}, \ldots, y_{0,i_1-1}), Q(1,1) = C_{i_1}(y_{1,0}, y_{1,1}, \ldots, y_{1,i_1-1}), \ldots, Q(1, j_1-1) = C_{i_1}(y_{j_1-1,0}, y_{j_1-1,1}, \ldots, y_{j_1-1,i_1-1})$ be the cycles of type H_1 in $M_1(i_1, j_1, k_1)$. Let $G_1(1, (2s+1)(mod\, j_1)) = \{x_{(2s+1)(mod\, j_1),0}, x_{(2s+1)(mod\, j_1),1}, \ldots, x_{(2s+1)(mod\, j_1),(i_1-3)/3}\}$ be the set of vertices which lie between horizontal cycles $Q(1, (2s+1)(mod\, j_1))$ and $Q(1, (2s+2)(mod\, j_1))$ for $0 \le s \le j_1 - 1$. Similarly, let $Q(2,0) = C_{i_2}(z_{0,0}, z_{0,1}, \ldots, z_{0,i_2-1}), Q(2,1) = C_{i_2}(z_{1,0}, z_{1,1}, \ldots, z_{1,i_2-1}), \ldots, Q(2, j_2-1) = C_{i_2}(z_{j_2-1,0}, z_{j_2-1,1}, \ldots, z_{j_2-1,i_2-1})$ be the cycles of type H_1 in $M_2(i_2, j_2, k_2)$ and $G_2(2, (2s+1)(mod\, j_2)) = \{w_{(2s+1)(mod\, j_2),0}, w_{(2s+1)(mod\, j_2),1}, \ldots, w_{(2s+1)(mod\, j_2),(i_2-3)/3}\}$ be the set of vertices which lie between horizontal cycles $Q(1, (2s+1)(mod\, j_2))$ and $Q(1, (2s+2)(mod\, j_2))$ for $0 \le s \le j_2 - 1$. Now, we have the following cases.

Case 1: If $(i_1, j_1, k_1) = (i_2, j_2, k_2)$ then $i_1 = i_2$, $j_1 = j_2$, $k_1 = k_2$. Define $f : V(M_1(i_1, j_1, k_1)) \to V(M_2(i_2, j_2, k_2))$ such that $f(y_{g,h}) = z_{g,h}$ for $0 \le g \le j_1 - 1$ and $0 \le h \le i_1 - 1$ and $f(x_{(2s+1)(mod\, j_1),h}) = w_{(2s+1)(mod\, j_1),h}$ for the vertices of $G_1(1, (2s+1)(mod\, j_1))$ and $G_2(2, (2s+1)(mod\, j_1))$ for all $0 \le s \le j_1 - 1$ and $0 \le h \le (i_1 - 3)/3$. By f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

Case 2(a): If $i_1 = i_2$, $j_1 = j_2 = 4m$ and $k_1 \neq k_2$, where $m \in \mathbb{N}$. We identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, \ldots, z_{j_2-1,0}, z_{0,k_2})$ and then cut along the path $P(z_{0,r}, z_{1,r-1}, z_{2,r-2}, z_{3,r}, x_{3,r/3}, \ldots, z_{0,r+k_2})$ for some $r = 3m$, where $m \in \mathbb{N}$ and $0 \le r \le i_2 - 1$. This gives a representation R of M_2 with a map $f' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f'(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2))}, & \text{if } 0 \le h \le i_2 - 1 \text{ and } 0 \le g \le j_2 - 1,\ g = 4m-1, 4m; \\ & \text{where } m \in \mathbb{N} \cup \{0\} \\ z_{g,(i_2+r-h-1)(mod\, i_2))}, & \text{if } 0 \le h \le i_2 - 1 \text{ and } 0 \le g \le j_2 - 1,\ g \neq 4m-1, 4m; \\ & \text{where } m \in \mathbb{N} \cup \{0\} \end{cases}$$

$$f'(w_{(2s+1)(mod\, j_2),h}) = \begin{cases} w_{(2s+1)(mod\, j_2),(i_2+r-3-3h)/3(mod\, i_2/3)}, & \text{if } 0 \le h \le (i_2-3)/3,\ s = 2m, \\ & 0 \le s \le j_2 - 1; \text{ where } m \in \mathbb{N} \cup \{0\} \\ w_{(2s+1)(mod\, j_2),(i_2+r-3h)/3(mod\, i_2/3)}, & \text{if } 0 \le h \le (i_2-3)/3,\ s = 2m+1, \\ & 0 \le s \le j_2 - 1; \text{ where } m \in \mathbb{N} \cup \{0\} \end{cases}$$

In R the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \ldots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \ldots, z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2}, z_{0,r+k_2-1}, \ldots, z_{0,r+k_2+1})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \ldots, z_{0,r+k_2})$ in Q' has length $r + i_2 - (r + k_2) = i_2 - k_2$. Note that, R has j_2 number of horizontal cycles of length i_2. So, we have $R = M(i_2, j_2, i_2 - k_2)$. By $l_{1,3} = l_{2,3}$, we see length$(Q_{1,3})$ = length$(Q_{2,3})$. This implies, $\min\{j_1 + j_1/4 + k_1, j_1 + j_1/4 + (i_1 - k_1)\} = \min\{j_2 + j_2/4 + (i_2 - k_2), j_2 + j_2/4 + i_2 - (i_2 - k_2)\}$. Since $i_1 = i_2, j_1 = j_2$ and $k_1 \neq k_2$, it follows that $k_1 + 5j_1/4 \neq k_2 + 5j_2/4$ and $j_1 + j_1/4 + (i_1 - k_1) \neq j_2 + j_2/4 + (i_2 - k_2)$. This gives $k_1 + 5j_1/4 = i_2 + 5j_2/4 - k_2 = i_1 + 5j_1/4 - k_2$ as $i_1 = i_2$ and $j_1 = j_2$. That is, $k_2 = i_1 - k_1$. Now $i_2 - k_2 = i_2 - (i_1 - k_1) = k_1$ since $i_2 = i_1$. Thus, $M_2(i_2, j_2, i_2 - k_2) = M_1(i_1, j_1, k_1)$. Therefore, by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

(b): If $i_1 = i_2$, $j_1 = j_2 = 4m + 2$ and $k_1 \neq k_2$, where $m \in \mathbb{N}$. We identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, \ldots, z_{j_2-1,0}, z_{0,k_2})$ of $M_2(i_2, j_2, k_2)$ and then cut along the path $P(z_{0,r}, z_{1,r-1}, z_{2,r-2}, z_{3,r}, x_{3,r/3}, \ldots, z_{0,r+k_2-1})$ for some $r = 3m$, where $m \in \mathbb{N}$ and $0 \le r \le i_2 - 1$. This gives representation R of M_2 with a map $f'' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f''(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2))}, & \text{if } 0 \le h \le i_2 - 1 \text{ and } 0 \le g \le j_2 - 1,\ g = 4m, 4m-1; \\ & \text{where } m \in \mathbb{N} \cup \{0\} \\ z_{g,(i_2+r-h-1)(mod\, i_2))}, & \text{if } 0 \le h \le i_2 - 1 \text{ and } 0 \le g \le j_2 - 1,\ g \neq 4m, 4m-1; \\ & \text{where } m \in \mathbb{N} \cup \{0\} \end{cases}$$

$$f''(w_{(2s+1)(mod\, j_2),h}) = \begin{cases} w_{(2s+1)(mod\, j_2),(i_2+r-3-3h)/3(mod\, i_2/3)}, & \text{if } 0 \le h \le (i_2-3)/3,\ s=2m, \\ & 0 \le s \le j_2-1; \text{ where } m \in \mathbb{N}\cup\{0\} \\ w_{(2s+1)(mod\, j_2),(i_2+r-3h)/3(mod\, i_2/3)}, & \text{if } 0 \le h \le (i_2-3)/3,\ s=2m+1, \\ & 0 \le s \le j_2-1; \text{ where } m \in \mathbb{N}\cup\{0\} \end{cases}$$

In R the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \ldots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \ldots, z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2-1}, z_{0,r+k_2-2}, \ldots, z_{0,r+k_2})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \ldots, z_{0,r+k_2-1})$ in Q' has length $r + i_2 - (r + k_2 - 1) = i_2 - k_2 + 1$. Note that, R has j_2 number of horizontal cycle of length i_2. So, we have $R = M_2(i_2, j_2, i_2 - k_2 + 1)$. By $l_{1,3} = l_{2,3}$, we see that length$(Q_{1,3})$ = length$(Q_{2,3})$. This implies, $\min\{j_1 + \lfloor j_1/3\rfloor + (k_1 - 1), j_1 + \lfloor j_1/3\rfloor + (i_1 - k_1 + 1)\}$ $= \min\{j_2 + \lfloor j_2/3\rfloor + (i_2 - k_2 + 1) - 1, j_2 + \lfloor j_2/3\rfloor + i_2 - (i_2 - k_2)\}$. If $k_1 + j_1 + \lfloor j_1/3\rfloor - 1 = k_2 + j_2 + \lfloor j_2/3\rfloor$, it implies that $k_1 - k_2 = 1$ since $j_1 = j_2$. This is not possible since $k_1 - k_2 = 3m$, where $m \in \mathbb{N}$. This gives, $k_1 + j_1 + \lfloor j_1/3\rfloor - 1 = i_2 + j_2 + \lfloor j_2/3\rfloor - k_2 = i_2 + j_1 + \lfloor j_1/3\rfloor - k_2$ as $j_1 = j_2$. That is, $k_1 = i_2 - k_2 + 1$. Thus, $M_2(i_2, j_2, i_2 - k_2 + 1) = M_1(i_1, j_1, k_1)$. Therefore, by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$.

(c): If $i_1 = i_2$, $j_1 = j_2 = 2$ and $k_1 \neq k_2$. In this case identify $M_2(i_2, j_2, k_2)$ along the vertical boundary $P(z_{0,0}, z_{1,0}, z_{0,k_2})$ of $M_2(i_2, j_2, k_2)$ and then cut along the path $P(z_{0,r}, z_{1,r}, z_{0,k_2+r-1})$ for some $r = 3m$, where $m \in \mathbb{N}$ and $0 \le r \le i_2 - 1$. Thus, we obtained a representation R of M_2 with a map $f''' : V(M_2(i_2, j_2, k_2)) \to V(R)$ such that

$$f'''(z_{g,h}) = \begin{cases} z_{g,(i_2+r-h)(mod\, i_2))}, & \text{if } g = 0 \text{ and } 0 \le h \le i_2 - 1 \\ z_{g,(i_2+r-h-1)(mod\, i_2))}, & \text{if } g = 1 \text{ and } 0 \le h \le i_2 - 1 \end{cases}$$

$f'''(w_{1,h}) = w_{1,(i_2+r-3-3h)/3(mod\, i_2/3)}$ for $0 \le h \le (i_2 - 3)/3$.

In R, the lower and upper horizontal cycles are $Q' = C_{i_2}(z_{0,r}, z_{0,r-1}, \ldots, z_{0,i_2-1}, z_{0,0}, z_{0,1}, \ldots, z_{0,r+1})$ and $Q'' = C_{i_2}(z_{0,r+k_2-1}, z_{0,r+k_2-2}, \ldots, z_{0,r+k_2})$ respectively. The path $P(z_{0,r}, z_{0,r-1}, \ldots, z_{0,r+k_2-1})$ in Q' has length $r + i_2 - (r + k_2 - 1) = i_2 - k_2 + 1$. In this process, R has j_2 number of horizontal cycle of length i_2. So, we have $R = M_2(i_2, j_2, i_2 - k_2 + 1)$. By assumption, $l_{1,3} = l_{2,3}$, length$(Q_{1,3})$ = length$(Q_{1,3})$. This implies, $\min\{j_1 + j_1/2 + (k_1 - 1), j_1 + j_1/2 + (i_1 - k_1 + 1)\}$ = $\min\{j_2 + j_2/2 + (i_2 - k_2 + 1) - 1, j_2 + j_2/2 + i_2 - (i_2 - k_2)\}$. If $k_1 + 3j_1/2 - 1 = k_2 + 3j_2/2$, then $k_1 - k_2 = 1$ since $j_1 = j_2$. This is not possible as $k_1 - k_2 = 3m$, where $m \in \mathbb{N}$. This gives, $k_1 + 3j_1/2 - 1 = i_2 + 3j_2/2 - k_2 = i_2 + 3j_1/2 - k_2$ as $j_1 = j_2$. That is, $k_1 = i_2 - k_2 + 1$. Thus, $M(i_2, j_2, i_2 - k_2 + 1) = M(i_1, j_1, k_1)$. Therefore, by f, $M_1(i_1, j_1, k_1) \cong M_2(i_2, j_2, k_2)$. So, the claim follows. Hence, $M_1 \cong M_2$.

Converse of the above lemma follows from the converse part of Lemma 3.5.2. □

Now doing the computing for the first four admissible values of $|V(M)|$, we get Table 3.7. For $|V(M)| = 28$, the computations is illustrated as follows:

Example 3.7.1 Let M be a DSEM of type $[3^6 : 3^2.4.3.4]$ with 28 vertices on the torus. By Lemma 3.7.1, M has four $M(i,j,k)$ representation, namely, $M(6,4,0), M(6,4,3), M(12,2,5)$, and $M(12,2,8)$ see Figures 3.7.3, 3.7.4, 3.7.5, 3.7.6 respectively. In $M(6,4,0)$, $Q_{1,1} = C_6(u_1, u_2, u_3, u_4, u_5, u_6)$ and $Q_{1,2} = C_5(u_1, v_1, x_1, y_1, z_1)$ are cycles of type H_1, H_2 respectively and $Q_{1,3} = C_5(u_1, v_1, x_1, y_1, z_1)$ is a cycle of type H_3. In $M(6,4,3)$, $Q_{2,1} = C_6(u_1, u_2, u_3, u_4, u_5, u_6)$, $Q_{2,2} = C_{10}(u_1, v_1, x_1, y_1, z_1, u_4, v_4, x_4, y_4, z_2)$, and $Q_{2,3} = C_8(u_4, v_4, x_4, y_4, z_2, u_1, u_2, u_3)$ are cycles of type H_1, H_2, H_3 respectively. In $M(12,2,5)$, $Q_{3,1} = C_{12}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$, $Q_{3,2} = C_{20}(x_1, y_1, x_6, y_5, z_2, x_{10}, y_{10}, x_3, y_2, z_1, x_7, y_7, x_{12}, y_{11}, z_4, x_4, y_4, x_9, y_8, z_3)$, and $Q_{3,3} = C_7(x_6, y_5, z_2, x_{10}, x_9, x_8, x_7)$ are cycles of type H_1, H_2, H_3 respectively. In $M(12,2,8)$, $Q_{4,1} = C_{12}(x_1, x_2, x_3, x_4, x_5, x_6, x_7,$

$x_8, x_9, x_{10}, x_{11}, x_{12})$, $Q_{4,2} = C_{20}(x_1, y_1, x_9, y_8, z_3, x_4, y_4, x_{12}, y_{11}, z_4, x_7, y_7, x_3, y_2, z_1, x_{10}, y_{10}, x_6, y_5, z_2)$, and $Q_{4,3} = C_8(x_9, y_8, z_3, x_4, x_5, x_6, x_7, x_8)$ are cycles of type H_1, H_2, H_3 respectively.

Since length$(Q_{1,1}) \neq$ length$(Q_{r,1})$ and length$(Q_{2,1}) \neq$ length$(Q_{r,1})$ for $r = 3, 4$, $M(6,4,0) \ncong M(12,2,5), M(12,2,8)$ and $M(6,4,3) \ncong M(12,2,5), M(12,2,8)$. Also, length$(Q_{1,2}) \neq$ length$(Q_{2,2})$, $M(6,4,0) \ncong M(6,4,3)$. We identify $M(12,2,5)$ along the path $P(x_1, y_1, x_6)$ and cut along the path $P(x_4, y_3, x_8)$. This gives $M(12,2,8)$ represntation, given in Figure 3.7.7. By the isomorphism map define in Lemma 3.7.2, $M(12,2,5) \cong M(12,2,8)$. Therefore, there are three DSEMs of type $[3^6 : 3^2.4.3.4]$ with 28 vertices on the torus upto isomorphism.

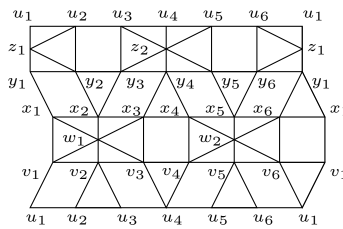

Figure 3.7.3: $M(6,4,0)$

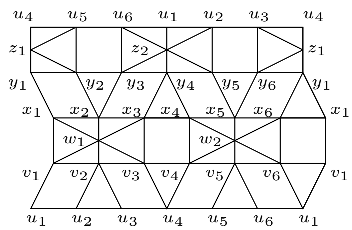

Figure 3.7.4: $M(6,4,3)$

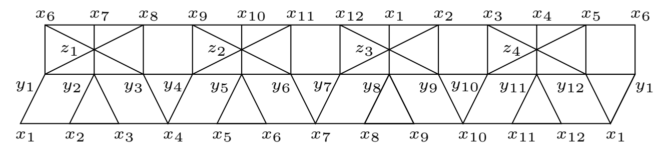

Figure 3.7.5: $M(12,2,5)$

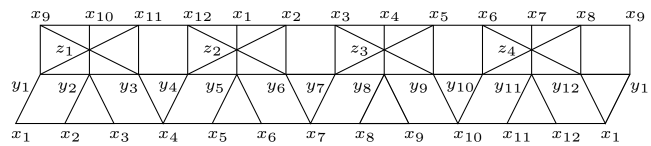

Figure 3.7.6: $M(12,2,8)$

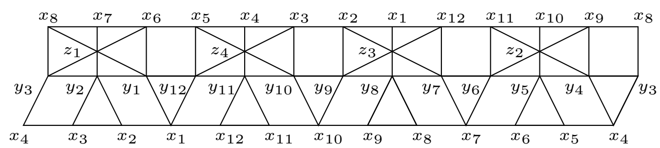

Figure 3.7.7: $M(12,2,8)$

Table 3.7: DSEMs of type $[3^6 : 3^2.4.3.4]$ on the torus for $|V(M)| \leq 42$

$\lvert V(M)\rvert$	Isomorphic classes	Length of cycles	No of maps
21	$M(9,2,5)$	$(9,5,7)$	1
28	$M(6,4,0)$	$(6,5,5)$	3
	$M(6,4,3)$	$(6,10,8)$	
	$M(12,2,5), M(12,2,8)$	$(12,20,7)$	
35	$M(15,2,5), M(15,2,11)$	$(15,25,7)$	2
	$M(15,2,8)$	$(15,5,10)$	
42	$M(6,6,2), M(6,6,5)$	$(6,30,9)$	5
	$M(9,4,0)$	$(9,5,5)$	
	$M(9,4,3), M(9,4,6)$	$(9,15,8)$	
	$M(18,2,5), M(18,2,14)$	$(18,10,7)$	
	$M(18,2,8), M(18,2,11)$	$(18,30,10)$	

Proof of theorem 2.1. Proof of the Theorem 2.1 follows from the results of the Sections 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7. □

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